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The calculation of Feynman diagrams in the superstring perturbation theory

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Abstract

The method of the perturbative calculation of the multi-loop amplitudes in the superstring theories is proposed. In this method the multi-loop superstring amplitudes are calculated from the equations that are none other than Ward identities. The above equations are derived from the requirement that the discussed amplitudes are independent from a choice of gauge of both the *vierbein* and the gravitino field. The amplitudes in question are determined in the unique way by these equations together with the factorization condition on the multi-loop amplitudes when two handles move away from each other. The considered amplitudes are calculated in the terms of vacuum correlators of superfields defined on the complex $(1|1)$ supermanifolds. The above supermanifolds are described by superconformal versions of Schottky groups. The superconformal Schottky groups appropriate for this aim are built for all the spinor structures.

Being based only on the gauge invariance together with the factorization requirement on the multi-loop amplitudes when the handles move away from each other (the unitarity), the proposed method can be used widely in the critical (super)string theories. Moreover, after an appropriate modification to be made, this method can be employed for non-critical (super)string, too. In this paper the closed, oriented Ramond-Neveu-Schwarz superstring is considered, only boson emission amplitudes being discussed. The problem of the calculation of the multi-loop boson emission amplitudes is concentrated, in mainly, on those spinor structures where superfields is branched on the complex z-plane where Riemann surfaces are mapped. In this case the vacuum superfield correlators can not be derived by a simple extension of the boson string results. The method of the calculation of the above correlators is proposed. The multi-loop amplitudes associated with all the even spinor structures are calculated in the explicit form. A previous discussion of the divergency problem is given.

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1 Introduction

Superstrings [1-4] currently are the only candidates for a basis of the unified theory including gravity [5]. Nevertheless, superstrings still require a long way to be understood enough even in the framework of the perturbation theory.

It seems that superstring models do not contain an enormous strong high energy gravity interaction, which appears in other gravity theories making them to be non-renormalizable. Moreover, the one-loop calculations [4,6] encourages a hope that the unified theory based on superstrings might be finite. The problem arises, however, whether the one-loop approximation results can be extended to all orders of the perturbation theory. The more so, additional divergences could appear in multi-loop superstring amplitudes, just as they do appear [7] in the boson string theory. These divergences are due to a degeneration of genus-n Riemann surfaces ($n > 1$) into a few ones of the lower genus. The essential progress in the investigation of this problem was achieved [8] recently, but it seems to be desirable to continue the study of the above problem in different superstring models.

Furthermore, two essentially different superstring schemes are presently discussed [4], they being the manifestly space-time (10-dim.) supersymmetrical Green-Schwarz scheme [3] and the manifestly world-sheet supersymmetrical Ramond-Neveu-Schwarz one [1]. It is generally believed, however, that, after the GSO projection [2], the Ramond-Neveu-Schwarz superstrings also possess a hidden space-time supersymmetry and, therefore, both the above schemes correspond to the same physical model. Till now, however, the general proof [4] of this statement exists on a quite formal level, the direct proof being done in the tree and one-loop approximations only [4]. To prove the above statement for multi-loop amplitudes one should verify the validness of the non-renormalization theorems [4,9].

The study of the discussed problems requires calculations of the multi-loop amplitudes in question. Besides, after the multi-loop amplitudes being calculated, other significant goals could be outlined, for example, the perturbative calculation of ultraviolet and infrared asymptotics of superstring amplitudes. It might stimulate new ideas beyond the superstring perturbation theory.

In this paper we present the calculation of the above multi-loop amplitudes. We employ the method of the multi-loop calculations in (super)string theories, which has been proposed in [10-12]. The considered method allows to obtain the multi-loop amplitudes in the form appreciable for the investigation of the divergency problem and of different asymptotics, as well. Being based only on the gauge invariance together with the factorization requirement on the multi-loop amplitudes when the handles move away from each other (the unitarity), this method can be used widely in the critical (super)string theories. Moreover, after an appropriate modification to be made, the above method can be employed for non-critical (super)string, too. But in this paper we concentrate on the closed, oriented Ramond-Neveu-Schwarz superstring, only boson emission amplitudes being considered. We calculate in the explicit form the multi-loop amplitudes associated with all the even spinor structures. At last, we touch the divergency problem.

The multi-loop calculations in the superstring theory are discussed already for a long time. In the well known scheme [13-16] the multi-loop Ramond-Neveu-Schwarz amplitudes are written

to be sums over ordinary spin structures [17] integrated over Riemann moduli. The above amplitudes are usually constructed [13,15,16,18,19] in the terms of suitable modular forms. In this approach, however, for more than three loop amplitudes, one is forced to use complicated sets of moduli that prevents the study of the amplitudes obtained [20]. Moreover, in this scheme multi-loop amplitudes appear to be depended on a choice of basis of the gravitino zero modes [13,15,16]. It means that the two-dimensional supersymmetry is lost in the scheme discussed. Indeed, in the superstring theory both the "*vierbein*" and the gravitino field are the gauge fields. Owing to the gauge invariance the "true" superstring amplitudes are independent of a choice of a gauge of the above gauge fields. Therefore, they have no dependence on a choice of basis of the gravitino zero modes.

The discussed dependence on a choice of basis of the gravitino zero modes appears to be a serious difficulty in the considered scheme. But one can hope that the above difficulty is absent in the formulation [21] possessing of the manifest two-dimensional supersymmetry. It is the formulation, that is used in the present paper. In this case the multi-loop superstring amplitudes are obtained [22,23] (see also [11,12]) by the summation over "superspin" structures integrated over both the even moduli and the odd ones. The above superspin structures are defined for superfields on the complex $(1|1)$ supermanifolds [21]. They present supersymmetrical versions of the ordinary spin structures on Riemann surfaces. Being twisted about (A,B)-cycles, the superfields are changed by mappings that present superconformal versions of fractional linear transformations. Generally, every considered mapping depends on $(3|2)$ parameters [21]. For odd parameters to be arbitrary, the above mappings include, in addition, fermion-boson mixings. It differs the superspin structures from the ordinary spin ones. Indeed, the ordinary spin structures [17] imply that boson fields are single-valued on Riemann surfaces. Only fermion fields being twisted about (A,B)-cycles may receive the factor (-1). For all odd parameters to be equal to zero every genus-n superspin structure $L=(l_1,l_2)$ is reduced to the ordinary (l_1,l_2) spin one. Here l_1 and l_2 are the theta function characteristics: $(l_1,l_2)=\bigcup_s(l_{1s},l_{2s})$ where $l_{is}\in(0,1/2)$. The (super)spin structure is even, if $4l_1l_2=4\sum_{s=1}^n l_{1s}l_{2s}$ is even. It is odd, if $4l_1l_2$ is odd. For the discussed superspin structures it is convenient to use superstring analogues of the Schottky groups [24,25]. Apparently, it is the only modular parameterization that allows to perform explicit calculations of the partition functions in the terms of the even and odd moduli. In the discussed parameterization fermion fields are periodical about the A_s-cycle only, if $l_{1s}=0$. In the $l_{1s}\neq 0$ case the fermion fields are non-periodical about the A_s-cycle, superfields being branched on the complex z-plane where Riemann surfaces are mapped.

In the critical superstring theory the problem of the calculation of the multi-loop boson emission amplitudes is concentrated, in mainly, on those superspin structures where at least one of the l_{1s} characteristics is unequal to zero. Indeed, for superspin structures where all the l_{1s} characteristics are equal to zero, the multi-loop amplitudes can be derived [22] by a simple extension of the boson string results [26]. All the other superspin structures can not be derived in this way. Generally, the procedure of "sewing" [23] allows to consider the discussed superspin structures, but this scheme seems to be complicated, the results being obtained in the form that is rather difficult for an investigation.

In the superstring theory the problem of supersymmetrization of the ordinary spin structures arises. Generally, there are different ways to supersymmetrize ordinary spin structures, but do

not all supersymmetrizations appear to be appropriated for the superstring theory. Especially, because the space of half-forms does not necessarily have a basis when there are odd moduli [27]. Besides, the chosen set of moduli is due to be appropriate for constructing of supermodular invariant superstring amplitudes [11,12]. The super-Schottky groups suitable for the superstring theory have been constructed in [28-30]. For the l_{1s} characteristic to be equal to zero the super-Schottky groups have been also built in earlier papers [14,22,23].

In the case when all the l_{1s} characteristics are equal to zero, the superfield vacuum correlators have been derived [11,12,22] by a simple extension of the boson string correlators [10,26]. In the opposite case when at least one of the l_{1s} characteristics are unequal to zero, the superfield vacuum correlators cannot be derived directly from the boson string theory. The method of calculating the vacuum superfield correlators assigned to the discussed superspin structures has been proposed in [28,30]. In details this method is developed in the present paper where we calculate the vacuum superfield correlators for the discussed even superspin structures. Together with the results obtained in [11,12,22] the results of this paper give the vacuum correlators for all the even superspin structures. The above correlators are used for the calculation of the superstring amplitudes by the method developed in this paper.

This method is based on the path-integral formulation [31] of (super)string theories. This formulation allows to employ widely the local gauge symmetries of the (super)string. In this case a considerable understanding of (super)string theories has been reached [25] already in the framework of both the (super)conformal gauge symmetry and the BRST invariant quantization procedure. But in the above approach even for the multi-loop boson string amplitudes it is failed to find [14] factors due to the moduli volume form and the ghost zero modes, as well. So in the framework of the discussed approach one can hardly hope to study satisfactory those quite complicated spin structures where fermion fields appear to be non-periodical about even if the only A-cycle. The above spin structures can be studied in the approach to (super)string theory developed in [10-12,28]. This method employs widely not only the (super)conformal symmetry, but all the local gauge symmetries of the (super)string. Besides, the presented method employs neither BRST quantization nor the bosonization prescription.

In the proposed method [10-12,28] the multi-loop amplitudes are calculated from equations that are none other than Ward identities. The above equations are obtained from the condition that the discussed amplitudes are independent of a choice of the gauge fields, for superstrings they being the *vierbein* and the gravitino field. In particular, multi-loop amplitudes appear to be independent of a choice of basis of the gravitino zero modes.

The discussed equations are derived in the framework of the special ghost scheme [10-12] that allows to calculate both the moduli volume form and zero mode contributions by a suitable modification of the vacuum correlator of the ghost superfields. Unlike the usual ghost scheme [31], this scheme includes "global ghost" parameters, as well as ghost fields. Then, the gauge fields being fixed, the multi-loop amplitudes are given by integrals over the string and ghost fields together with these "global ghost" parameters. As far as the integrals over the above fields need ultra-violet regularization, the obtained expressions are used only to derive equations for the amplitudes in question.

The above equations resemble those discussed in [14,25]. But, unlike [14,25], these equations take into account, in addition, the factors due to both ghost zero modes and the moduli volume

form. It is urgent especially for those spin structures where fermion fields are non-periodical about A-cycles because in this case the equations given in [14,25] have no solutions at all. Besides, unlike the equations in [14,25], the discussed equations are (super)modular invariant. So, after a suitable summation over spin structures being performed, one can be sure that superstring amplitudes satisfy restrictions due to the modular invariance though a direct proof of this statement may be quite difficult.

Being differential in moduli, the discussed equations determine the partition functions up to constant factors. To calculate all these factors in the terms of a coupling constant, the factorization requirement on the amplitudes is used when two handles move away from each other. This requirement replaces the unitarity equations. Though the vertices are known a long time already, it seems interesting to note that they could be calculated in this way, too. So, the amplitudes turn out to be fully determined by the gauge invariance together with the "factorization requirement" above.

For the closed, critical boson string this method gives [10] the partition functions to be the same, as in [26]. This approach has been also applied [11,12] to calculating the multi-loop boson emission amplitudes of closed, oriented critical Ramond-Neveu-Schwarz superstrings, superspin structures corresponding to the fermion fields periodical about all A-cycles being considered. For the even superspin structures where at least one of the l_{1s} characteristics is unequal to zero, the partition functions were previously considered in [28]. The above partition functions were found to be much more complicate than those obtained [14,23] by a "naive" extension of the genus-1 ones. In details the calculation of the discussed partition functions is considered in the present paper.

This paper is organized as it follows. To explain the method we consider in Sec.II the closed, critical boson string, the superstring specification being ignored for the moment. In Sec.III we give the equations for multi-loop amplitudes in the superstring theory. In mainly, the above results have been obtained early [10-12,29], but they are given in the above Sections because it is necessary for understanding the following Sections IV-VII. In Sec.IV we calculate the vacuum correlator of the scalar superfields for the even superspin structures with at least one of the l_{1s} characteristics is unequal to zero. Also, in this case we calculate half-forms and the period matrices assigned to the supermanifolds [21,32]. In Sec.V we calculate the vacuum correlator of the ghost superfields for the above discussed even superspin structures. In Sec.VI the formulae for the multi-loop amplitudes associated with even superspin structures are obtained. The final expressions for the discussed multi-loop amplitudes associated with all the even superpin structures are given in Section VII. In this Section we also touch the divergency problem in the closed, oriented superstring theory. Details of the calculations are given in the Appendices.

2 Equations for multi-loop boson string amplitudes

As it has been noted in Sec.I, we calculate the multi-loop amplitudes from the equations that are none other than Ward identities. We start with the multi-loop amplitudes given in the form of integrals [31] over the two-dimensional metrics $g^{\alpha\beta}$. In the boson string theory the above two-dimensional metrics are the gauge fields. To write the multi-loop amplitudes in the

form of integrals over these $g^{\alpha\beta}$ metrics we map Riemann surfaces on a complex plane F fixing both *kleinian* groups K_n and fundamental domains Ω_n to be the same for all genus-n surfaces [10]. Then the two-dimensional metrics $g^{\alpha\beta}$ could be chosen in an arbitrary form inside the fundamental domain Ω_n. All above $g^{\alpha\beta}$ can be reduce to the full set $\{\hat{g}^{\alpha\beta}\}$ of the reference metrics by gauge mappings ϕ that are isomorphisms: $K_n \xrightarrow{\phi} K_n$ and $\Omega_n \xrightarrow{\phi} \Omega_n$. This reduction is impossible within the set in question. In the explicit form [1]

$$g^{\alpha\beta}(F,\overline{F}) = \partial_\xi F^\alpha(u,\overline{u})\partial_\eta F^\beta(u,\overline{u})\hat{g}^{\xi\eta}(u,\overline{u},\{\hat{q}_N\},\{\overline{\hat{q}_N}\})\exp W(u,\overline{u}) \quad . \tag{1}$$

In addition to local complex coordinates u and their complex conjugated $\overline{u}$, the references metrics for $n \geq 2$ depend also on the set $\{\hat{q}_N\}$ of $3n-3$ complex parameters and their complex conjugated $\overline{\hat{q}_N}$, the above $\{\hat{q}_N\}$ set being defined modulo of the modular group. In other respects the references metrics are arbitrary. [2] Then every n-loop amplitude are rewritten to be the integral over both the string fields and the metrics $g^{\alpha\beta}$ divided by the (infinite) volume of the gauge group. There is no integration over Riemann moduli because *kleinian* groups are the same for all the genus-n surfaces. Employing eq.(1), one can write down $g^{\alpha\beta}$ in the terms of $\hat{g}^{\alpha\beta}$ (depending on $\{\hat{q}_N,\overline{\hat{q}_N}\}$), as well as the gauge functions F^α and W. It is worth-while to note that $g^{\alpha\beta}$ depends not only on functions F^α and W, but on parameters $(\hat{q}_N,\overline{\hat{q}_N})$, too. So, the integrals over $g^{\alpha\beta}$ turn into the ones over $(\hat{q}_N,\overline{\hat{q}_N})$, as well as both F^α and W. The region of the integrating over $(\hat{q}_N,\overline{\hat{q}_N})$ is determined by the modular invariance. Eq.(1) allows to compute the jacobian of the discussed transformation. After integrating over F^μ and W, this jacobian J can be represented by the integral over ghost fields c^μ and $b_{\alpha\beta} = b_{\beta\alpha}$ together with $3n-3$ complex Grassmann "global ghosts" $\hat{\kappa}_N$ and their complex conjugated the $\overline{\hat{\kappa}_N}$ as

$$J = \int e^{-S_{gh}} \prod \delta(\sqrt{-\hat{g}}b_{\alpha\beta}\hat{g}^{\alpha\beta})(Db_{\xi\eta}Dc^\nu)\prod_{s=1}^{3n-3} d\hat{\kappa}_N d\overline{\hat{\kappa}_N} \quad . \tag{2}$$

In (2) the ghost action S_{gh} is given by

$$S_{gh} = -\int b_{\alpha\beta}\left[P_\mu^{\alpha\beta}c^\mu + \frac{\partial\hat{g}^{\alpha\beta}}{\partial\hat{q}_s}\hat{\kappa}_s\right]\sqrt{-\hat{g}}dud\overline{u} \tag{3}$$

where, in addition to repeated Greek indexes, the summation over s is implied. Moreover, $\hat{q}_s = (\tilde{q}_N,\overline{\hat{q}_N})$ and $\{\hat{\kappa}_s\} = \{\hat{\kappa}_N,\overline{\hat{\kappa}_N}\}$. Besides, $P_\mu^{\alpha\beta}$ is the well known [25] differential operator:

$$P_\mu^{\alpha\beta} = \hat{g}^{\alpha\nu}\partial_\nu\delta_\mu^\beta + \hat{g}^{\beta\nu}\partial_\nu\delta_\mu^\alpha - (-\hat{g})^{-1/2}\hat{g}^{\alpha\beta}\partial_\mu\sqrt{-\hat{g}}\hat{g}^{\alpha\beta} \quad . \tag{4}$$

One can see from (4) that $\hat{g}_{\alpha\beta}P_\mu^{\alpha\beta} = 0$. It is known [31] that volume form $(D\hat{b}_{\xi\eta}D\hat{c}^\nu)$ in eq.(2) depends on $\hat{g} = \det \hat{g}_{\alpha\beta}$, but it will be unessential for deriving the equations discussed.

[1] Throughout this Section the summation over Greek indexes repeated twice is implied.

[2] To every set $\{\hat{q}_N\}$ one can assign the set of Riemann moduli. As an example, one can map all the genus-1 surfaces on the rectangle $(1, ia)$ with $a > 0$, a being the same for all the genus-1 surfaces. Then $\hat{g}^{\alpha\beta} = \hat{g}^{\alpha\beta}(u,\overline{u},q,\overline{q})$ where q is a complex parameter. If one reduce $\hat{g}^{\alpha\beta}$ to the plane form ($\hat{g}^{\alpha\beta} \to \delta^{\alpha\beta}$) then the surfaces above turn out to be mapped on quadrangles $(1,\omega)$ with $\omega = \omega(q,\overline{q})$.

The "global ghosts" $\hat{\kappa}_s$ are the peculiarity of the presented scheme that differs this scheme from that developed in [25]. It must be stressed that in this scheme the integrating is performed over all modes of the ghost fields including $b_{\alpha\beta}$-zero modes. The integral over the above zero modes appears to be finite owing to the proportional to $\hat{\kappa}_s$ terms in the ghost action (3). As a result every n-loop amplitude $A_n^{(b)}$ can be written as the integral over the string fields and the ghost fields together with the global Grassmann parameters $\hat{\kappa}_s$. But the integrals over both the ghost and string fields need ultra-violet regularization that hampers a direct calculation of n-loop amplitudes $A_n^{(b)}$. So we use the obtained expression for $A_n^{(b)}$ only to derive Ward identities

$$\delta_\perp A_n^{(b)} = 0 \tag{5}$$

where $\delta_\perp A_n$ are alterations of $A_n^{(b)}$ caused by transverse infinitesimal arbitrary variations $\delta_\perp g_{\alpha\beta}$ of the references metrics $\hat{g}_{\alpha\beta}$ ($\hat{g}_{\alpha\beta}\delta_\perp g_{\alpha\beta} = 0$). Then we employ the above Ward identities to calculate $A_n^{(b)}$.

For this aim we reduce $\hat{g}^{\alpha\beta}$ to the conform plane form by a mapping $u \to z(u,\bar{u})$ on a new complex plane z. Simultaneously, $\hat{\kappa}_N \to \kappa_N(\{\kappa_M, \overline{\kappa_M}\})$ and $\hat{q}_N \to q_N(\{q_M, \overline{q_M}\})$, q_N being Riemann moduli. In this case the terms depending on $\hat{\kappa}_s$ in (3) can be included [10] in the vector ghost field $r(z,\bar{z})$ as

$$r(z,\overline{z}) = c(z,\overline{z}) - \sum_N \left(\frac{\partial z(u,\overline{u})}{\partial q_N}\right)_{u,\overline{u}} \kappa_N \tag{6}$$

where c is the vector conformal field. The proportional to κ_N terms in (6) are originated by the proportional to $\tilde{\kappa}_N$ terms in eq.(3). Then the ghost action (3) can be written in the terms of both the above r field and the tensor conformal ghost field $b(z,\bar{z})$ as

$$S_{gh} = -2\int (b\overline{\partial}r + \overline{b}\partial\overline{r})dzd\overline{z} \tag{7}$$

So, S_{gh} has the usual form [4,25] in the terms of both the tensor conformal ghost field b and the vector field r, but, unlike [4,25], the vector field in eq.(7) has depending on z periods [10] under *kleinian* group transformations $z \to g_s(z)$ associated with 2π-twists about B_s-cycles. Indeed, it can be proved [10] from (6) that

$$r(g_s(z),\overline{g}_s(z)) = r(z,\overline{z})\partial g_s(z) + \sum_N \left(\frac{\partial g_s(z)}{\partial q_N}\right)_z \kappa_N \quad . \tag{8}$$

Therefore, the ghost vacuum correlator

$$G(z,z') = - < r(z,\overline{z})b(z',\overline{z'}) > \tag{9}$$

also has periods on the z plane. In the explicit form [10]

$$G(g_s(z),z') = (\partial g_s(z))G(z,z') - \sum_N \left(\frac{\partial g_s(z)}{\partial q_N}\right)_z \hat{\chi}_N(z') \tag{10}$$

where $\hat{\chi}_N(z')$ are 2-tensor zero modes:

$$\hat{\chi}_N(z) = - < \kappa_N b(z,\overline{z}) > \quad . \tag{11}$$

To avoid misunderstands, it is useful to remind that in the discussed scheme the integration performs over all modes of the tensor ghost field including its zero modes. The integral over the above zero modes appears to be convergent owing to the proportional to κ_N terms in eq.(6). Furthermore, eqs.(10) together with the condition for $G(z,z')$ to be a conform 2-form on z'-plane determine both $G(z,z')$ and $\hat{\chi}_N(z')$ in the unique way [10]. Unlike the ghost correlator discussed in [14], the $G(z,z')$ ghost correlator satisfying (10) has no unphisical poles [10].

The n- loop amplitudes $A_n^{(b)}$ can be written as

$$A_n^{(b)} = \int Z_n^{(b)} < V > \prod_{N=1}^{3n-3} dq_N d\overline{q}_N \tag{12}$$

where $Z_n^{(b)}$ is the partition function and $< V >$ denotes the vacuum expectation of the vertex product. Then from eq.(5) one can obtain [10] the following equations for $Z_n^{(b)}$:

$$\sum_N \hat{\chi}_N(z) \frac{\partial}{\partial q_N} \ln Z_n^{(b)} = -\sum_N \frac{\partial}{\partial q_N} \hat{\chi}_N(z) + < T_m^{(b)} + T_{gh}^{(b)} > \tag{13}$$

together with the complex conjugated to (13) ones. In eqs.(13) the $\hat{\chi}_N(z')$ tensor zero modes are the same as in (10). Furthermore, $T_{gh}^{(b)}$ and $T_m^{(b)}$ are the stress tensors of the ghost and string fields, respectively:

$$T_m^{(b)} = -\frac{1}{2}\partial X^M \partial X_M \quad \text{and} \quad T_{gh}^{(b)} = 2b\partial r + (\partial b)r \quad , \tag{14}$$

X^M being the string fields; $M = 1, 2, ..., d$ where $d = 26$. One can see that both $T_{gh}^{(b)}$ and $T_m^{(b)}$ have the usual form [25] in the terms of the ghost or string fields, but $T_{gh}^{(b)}$ is calculated with the ghost vacuum correlator, which obey eqs.(10) instead of that discussed in [14]. So in the considered scheme the value $T_m^{(b)} + T_{gh}^{(b)}$ is not conformal 2-form under *kleinian* group mappings. But the right side of eq.(13), as well as the left side, appears to be 2-form under the mapping above. Moreover, it can be prove that eq.(13) is invariant under modular transformations $z \to \tilde{z}(z, \{q_N\})$ with the simultaneous change $q_N \to \tilde{q}_N(\{q_M\})$, $\tilde{q}_N$ being new moduli. Furthermore, under the discussed transformations a number of rounds about (A, B) cycles corresponds to every 2π-twist about B_s-cycle. Therefore, to every $z \to g_s(z)$ mapping one can assign the mapping $\tilde{z} \to \tilde{g}_{(s)}(\tilde{z})$, which describes the above rounds. So $\tilde{z}(g_s(z)) = \tilde{g}_{(s)}(\tilde{z})$. The modular transformations $G(z,z') \to \tilde{G}(\tilde{z},\tilde{z}')$ of the ghost vacuum correlator $G(z,z')$ are determined from the requirement that $\tilde{G}(\tilde{z},\tilde{z}')$ changes under *kleinian* group mappings on $\tilde{z}$-plane in the accordance with eqs.(10) written in the terms of new variables. One can verify that eq.(13) remains invariant under the discussed transformation. Details of the proof of this statement are planned to give in an another paper.

Eqs.(13) have been solved in [10]. The resulting partition functions appear to be the same as in [26]. It proves the modular invariance of the discussed partition functions that has been not properly proved in [26]. In the next Section we extend the above equations (13) to the superstring theory.

3 Multi-loop Superstring Amplitudes

In the considered superstring theory the n-loop amplitudes A_n are given [22,23] (see also [11,12]) by the sums over "superspin" structures integrated over $(3n-3|2n-2)$ complex moduli q_N and their complex conjugated $\overline{q}_N$, as well:

$$A_n = \int \prod_N dq_N d\overline{q}_N \sum_{L,L'} \hat{Z}^{(n)}_{L,L'} < V >_{L,L'} \tag{15}$$

where $\hat{Z}^{(n)}_{L,L'}$ are the partition functions and $< V >_{L,L'}$ denote the vacuum expectations of the vertex products. The index L (L') labels "superspin" structures of right (left) fields. In this paper we discuss only the even superspin structures.

As it has been already noted in Sec. I, the above superspin structures are defined for superfields on the complex $(1|1)$ supermanifolds [21]. We map these supermanifolds by the supercoordinate $t = (z|\theta)$ where z is a local complex coordinate and θ is its odd partner. To every discussed supermanifold the period matrix can be assigned [21,32]. The above genus-n period matrices $\omega(\{q_N\}; L)$ present periods about B-cycles of holomorphic superfunctions forming a suitable basis on the considered supermanifold. It is worth-while to note that the discussed period matrices depend on the superspin structure in the terms proportional to odd moduli[23,28,30].

The q_N moduli in (15) are defined modulo the supermodular group presenting a supersymmetrical version of the modular one. For the considered theory to be self-consistent, the integrand in eq.(15) being multiplied by the product of the differentials of moduli must be invariant under transformations of the supermodular group.

Under the discussed transformations the t supercoordinate is changes by holomorphic supersymmetrical mappings [21,32]: $t \to \tilde{t}(t)$. Simultaneously, $q_N \to \tilde{q}_N(\{q_M\})$. Also, generally, the above transformations turn out the superspin structures into each other: $L \to \tilde{L}$. Supermodular transformations $\omega(\{q_N\}; L) \to \omega(\{\tilde{q}_N\}; \tilde{L})$ of the period matrices have the same form as the unimodular transformations of period matrices assigned to Riemann surfaces:

$$\omega(\{q_N\}; L) = [A\omega(\{\tilde{q}_N\}; \tilde{L}) + B][C\omega(\{\tilde{q}_N\}; \tilde{L}) + D]^{-1} \tag{16}$$

where A, B, C and D are integral matrices discussed in [9] (see also [12]). In the boson string theory eqs.(16) would determine in an implicit form all the new moduli $\{\tilde{q}\}$ in the terms of the old ones $\{q\}$ up to arbitrariness due to possible fractionally linear transformations of Riemann surfaces. To avoid misunderstands it is necessary to note that in the superstring theory eqs.(16) are insufficient to determine all the $\tilde{q}_N$ moduli in the terms of the q_N ones because of the presence of odd moduli.

Two supermanifolds are topological non-equivalent, if they can not be obtained from each other by a supermodular transformation. The region of the integration over even moduli in eq.(15) is determined by the condition that different varieties of q_N in (15) correspond to topological non-equivalent supermanifolds. It is similar to the boson string theory where the region of the integration over moduli is determined by the modular invariance. Because of the supermanifods are non-compact in the sense of ref. [33] the boundary Σ of the discussed region,

generally, depends on the odd moduli. The dependence on odd moduli in Σ must necessarily be taken into account in the integrating over the odd moduli in eq.(15).

Every superspin structure given on a genus-n complex (1|1) supermanifold is defined by the mappings $(\Gamma_{a,s}(l_{1s}), \Gamma_{b,s}(l_{2s}))$ that are associated with rounds about (A_s, B_s)-cycles, respectively $(s = 1, 2, ..., n)$. The above mappings present supersymmetrical versions of fractional linear transformations. As it has been noted in Sec. I, there are different ways to supersymmetrize the fractional linear transformations, but do not all supersymmetrizations appear to be appropriated for the description of the mapping discussed. Firstly, the space of half-forms is due to have a basis. Otherwise [27] one meets with difficulties in construction of the vacuum correlator of the scalar superfields. Besides, the chosen set of moduli is due to be convenient to distinguish in eq.(15) between the superspin srtuctures $(l_{2s} = 0, l_{2s} = 1/2)$ and $(l_{1s} = 0, l_{2s} = 0)$ (see the discussion following after eq.(19) of this paper). The $(\Gamma_{a,s}(l_{1s}), \Gamma_{b,s}(l_{2s}))$ mappings given below are conformed with these requirements.

Following [14,22,23] and [11,12] we use for $\Gamma_{b,s}(l_{2s})$ the superconformal versions of Schottky transformations. The above Schottky transformations are defined as

$$z \longrightarrow \frac{(a_s z + b_s)}{(c_s z + d_s)} \quad \text{with} \quad a_s d_s - b_s c_s = 1. \tag{17}$$

Simultaneously the θ spinor receives the $(c_s z + d_s)^{-1}$ factor. Moreover, for $l_{2s} = 0$, the spinors are multiplied by (-1). The discussed $\Gamma_{b,s}(l_{2s})$ present superconformal versions of the above transformations. For $l_2 = 1/2$ one can choose [11,12,13,22,23] the $\Gamma_{b,s}(l_{2s} = 1/2)$ mapping to be

$$\begin{aligned} z \longrightarrow \frac{a_s(z+\theta\varepsilon_s)+b_s}{c_s(z+\theta\varepsilon_s)+d_s}, \quad \theta \longrightarrow \frac{\theta+\varepsilon_s}{c_s(z+\theta\varepsilon_s)+d_s} \\ \text{with} \quad \varepsilon_s = \alpha_s(c_s z + d_s) + \beta_s, \qquad a_s d_s - b_s c_s = 1 - \varepsilon_s \partial_z \varepsilon_s \quad . \end{aligned} \tag{18}$$

In (18) the even (a_s, b_s, c_s, d_s) and odd (α_s, β_s) parameters can be expressed [12,13] in the terms of two fixed points $(u_s|\mu_s)$ and$(v_s|\nu_s)$ on the complex (1|1) supermanifold together with the multiplier k_s as

$$\begin{aligned} a = \frac{u - kv - \sqrt{k}\mu\nu}{\sqrt{k}(u - v - \mu\nu)}, \quad d = \frac{ku - v - \sqrt{k}\mu\nu}{\sqrt{k}(u - v - \mu\nu)}, \quad c = \frac{1-k}{\sqrt{k}(u - v - \mu\nu)}, \\ \alpha = (\mu + \sqrt{k}\nu)(1+\sqrt{k})^{-1}, \quad \beta = -(\nu + \sqrt{k}\mu)(1+\sqrt{k})^{-1}, \end{aligned} \tag{19}$$

the index s being omitted. We choose (3|2) of the (u_s, v_s, μ_s, ν_s) parameters to be the same for all the genus-n supermanifolds, the rest of them together with the k_s multipliers being $(3n-3|2n-2)$ complex moduli q_N in (15). Without loss of generality one can think that $|k_s| < 1$. For the isomorphism between (18) and (19) to be, we fix the branch of $\sqrt{k}_s$, for example, as $|arg k_S| \leq \pi$.

In fact, the $\arg k_s \to \arg k_s + 2\pi$ replacement presents the (super)modular transformation turning $(l_{1s} = 0, l_{2s} = 1/2)$ into $(l_{1s} = 0, l_{2s} = 0)$.To prove this statement it is sufficient to check it for the genus $n = 1$. For $n = 1$ the period ω is given by [14,25] $\omega = (2\pi i)^{-1} \ln k$. So,

we see that ω is turned into $\omega+1$ under the replacement discussed. Employing the explicit form of the theta functions, one can verify that this transformation of ω is accompanied by the replacement $(l_1=0,l_2=1/2)\to(l_1=0,l_2=0)$.

We obtain $\Gamma_{b,s}(l_{2s}=0)$ by the above replacement in (19). Then the $|\arg k_s|\le\pi$ condition provides in eq.(15) the separating of the $(l_{1s}=0,l_{2s}=1/2)$ and $(l_{1s}=0,l_{2s}=0)$ superspin structures from each other. It is worth-while to note that under an another choice of the moduli it might be difficult to separate between the above superspin structures in eq.(15) that should originate difficulties in the calculation of the superstring amplitudes. As example, one can consider the transformations given in [22], the (3|2) parameters from those in eqs.(8) and (9) of ref. [22] being taken to be moduli.

Every the transformation (17) turns the circle $\hat C_s^{(-)}$ into $\hat C_s^{(+)}$ where

$$\hat C_s^{(-)}=\{z:|c_sz+d_s|=1\}\quad\text{and}\quad \hat C_s^{(+)}=\{z:|-c_sz+a_s|=1\}. \tag{20}$$

The round about the $\hat C_s^{(-)}$ or $\hat C_s^{(+)}$ circle corresponds to 2π-twist about A_s-cycle. For $l_{1s}=1/2$ the spinor fields are multiplied by (-1) under the above round [14]. So, for $l_{1s}=1/2$ the spinors turn out to be branched on the complex z-plane. Therefore, 2π-twists about A_s-cycles are associated with the following $\Gamma_{a,s}^{(o)}(l_{1s})$ mappings:

$$\Gamma_{a,s}^{(o)}(l_{1s})=\{z\to z,\theta\to(-1)^{2l_{1s}}\theta\}. \tag{21}$$

The discussed $\Gamma_{a,s}(l_{1s})$ present superconformal versions of the above transformations (21). It is obvious from (21) that $\Gamma_{a,s}(l_{1s}=0)=I$, but the $\Gamma_{a,s}(l_{1s}=1/2)$ mapping appears to be non-trivial.

To extend the discussed mappings (21) to arbitrary odd moduli it is necessary to find a relation between odd parameters in $\Gamma_{a,s}(l_{1s}=1/2)$ and those in $\Gamma_{b,s}(l_{2s})$. For this aim we employ [29,30] that for genus $n=1$, there are no odd moduli. Indeed, the genus-1 amplitudes are obtained in the terms of ordinary spin structures [17]. Then, for every particular s, all the odd parameters in both $\Gamma_{a,s}(l_{1s})$ and $\Gamma_{b,s}(l_{2s})$ can be reduced to zero by a suitable transformation $\tilde\Gamma_s$, which is the same for both the above transformations:

$$\Gamma_{a,s}(l_{1s})=\tilde\Gamma_s^{-1}\Gamma_{a,s}^{(o)}(l_{1s})\tilde\Gamma_s,\qquad \Gamma_{b,s}(l_{2s})=\tilde\Gamma_s^{-1}\Gamma_{b,s}^{(o)}(l_{2s})\tilde\Gamma_s \tag{22}$$

where $\Gamma_{a,s}^{(o)}(l_{1s})$ are given by (21), $\Gamma_{b,s}^{(o)}(l_{2s})$ are equal to $\Gamma_{b,s}(l_{2s})$ at $\mu_s=\nu_s=0$; $\tilde\Gamma_s$, in addition, depends on (μ_s,ν_s). We choose $\tilde\Gamma_s$ as

$$\begin{aligned}\tilde\Gamma_s:\qquad & z\to z_s+\theta_s\tilde\varepsilon_s(z_s),\qquad \theta\to\theta_s(1+\tilde\varepsilon_s\tilde\varepsilon_s'/2)+\tilde\varepsilon_s(z_s);\\ \tilde\varepsilon_s'=\partial_z\tilde\varepsilon_s(z),\qquad & \tilde\varepsilon_s(z)=[\mu_s(z-v_s)-\nu_s(z-u_s)](u_s-v_s)^{-1}.\end{aligned} \tag{23}$$

Employing (23) one can prove that the transformations (22) remain to be fixed the supermanifold points $(u_s|\mu_s)$ and $(v_s|\nu_s)$. In (22) the $\Gamma_{b,s}(l_{2s})$ mappings appear to be as those discussed above, $\Gamma_{b,s}(l_{2s}=1/2)$ being given by (18). Also, one can see from (22) and (23) that $\Gamma_{a,s}(l_{1s}=1/2)$ is equal to $\Gamma_{b,s}(l_{2s}=0)$ calculated at $\sqrt{k_s}=-1$. In the explicit form

$$\Gamma_{a,s}(l_{1s}=1/2)=\{z\to z-2\theta\tilde\varepsilon_s(z),\quad \theta\to-\theta(1+2\tilde\varepsilon_s\tilde\varepsilon_s')+2\tilde\varepsilon_s(z)\}. \tag{24}$$

One can see also that, for every s, $\Gamma_{a,s}(l_{1s})$ commutate with $\Gamma_{b,s}(l_{2s})$. Besides,

$$\Gamma_{b,s}(l_{2s}=0)=\Gamma_{a,s}(l_{1s}=1/2)\Gamma_{b,s}(l_{2s}=1/2). \tag{25}$$

Every $\Gamma_{b,s}l_{2s}$ mapping transforms the "circle" $C_s^{(-)}$ into the $C_s^{(+)}$ one, the above "circles" being

$$C_s^{(-)}=\{z:|Q_{\Gamma_{b,s}}(t)|=1\},\quad C_s^{(+)}=\{z:|Q_{\Gamma_{b,s}^{-1}}(t)|=1\} \tag{26}$$

where $Q_{\Gamma_{b,s}}(t)$ is the factor, which the spinor derivative $D(t)$ receives under the $\Gamma_{b,s}l_{2s}$ mapping, $D(t)$ being defined as

$$D(t)=\theta\partial_z+\partial_\theta \quad . \tag{27}$$

In eq.(27) derivative ∂_θ is meant to be the "left" one. For an arbitrary superconformal mapping $\Gamma=\{t\to t_\Gamma=(z_\Gamma(t)|\theta_\Gamma(t))\}$ this factor $Q_\Gamma(t)$ is given by

$$Q_\Gamma^{-1}(t)=D(t)\theta_\Gamma(t)\quad;\quad D(t_\Gamma)=Q_\Gamma(t)D(t). \tag{28}$$

Eqs.(26) take into account in the convenient form the boson-fermion mixing under rounds about B-cycles. It is why we below prefer to use (26) instead of (20) in the consideration of integrals over $d\theta dz$. By definition, the exterior (interior) of "circles" (26) is the same as that of circles (20).

If $l_{1s}\neq 0$, the cut $\tilde{C}_s$ appears on the considered z-plane. One of its endcut points is placed inside the $C_s^{(-)}$ circle and the other endcut point is placed inside the $C_s^{(+)}$ one. Superconformal p-tensors $F_p(t)$ being considered, every $\Gamma_{a,s}(l_{1s}=1/2)$ transformation relates $F_p(t)$ with its value $F_p^{(s)}(t)$ obtained from $F_p(t)$ by 2π-twist about $\hat{C}_s^{(-)}$-circle (20). So, $F_p(t)$ is changed under the $\Gamma_{a,s}(l_{1s})=\{t\to t_s^a\}$ and $\Gamma_{b,s}=\{t\to t_s^b\}$ mappings as

$$F_p(t_s^a)=F_p^{(s)}(t)Q^p_{\Gamma_{a,s}}(t),\qquad F_p(t_s^b)=F_p(t)Q^p_{\Gamma_{b,s}}(t). \tag{29}$$

The having zero periods vacuum correlator

$$\hat{X}_{L,L'}(t,\overline{t};t',\overline{t}')=<X(t,\overline{t})X(t',\overline{t}')> \tag{30}$$

of two identical scalar superfields X can be written in the terms of the holomorphic Green function $R_L(t,t')$, its periods $J_r(t;L)$ and the period matrix $\omega(\{q_N\};L)=\{\omega_{sr}(L)\}$ as

$$\begin{aligned}\hat{X}_{L,L'}(t,\overline{t};t',\overline{t}')=\left(J_r(t;L)+\overline{J_r(t;L')}\right)[2\pi i(\overline{\omega}(L')-\omega(L))^{-1}]_{rs}\times\\ \left(J_s(t';L)+\overline{J_s(t';L')}\right)+R_L(t,t')+\overline{R_{L'}(t,t')}.\end{aligned} \tag{31}$$

In (31) an explicit dependence on the moduli is omitted. Below we also omit the explicit dependence on L and L' implying that $\omega\equiv\omega(\{q_N\};L),\omega_{sr}\equiv\omega_{sr}(L),J_r(t)\equiv J_r(t;L)$ and $R(t,t')\equiv R_L(t,t')$. The above $R(t,t')$ changes under the $\Gamma_{a,s}(l_{1s})=\{t\to t_s^a\}$ and $\Gamma_{b,s}=\{t\to t_s^b\}$ mappings as

$$R(t_s^a,t')=R^{(s)}(t,t'),\qquad R(t_s^b,t')=R(t,t')+J_s(t')\quad . \tag{32}$$

Besides,

$$R(t,t') = R(t',t) \pm \pi i \tag{33}$$

We normalize $R(t,t')$ by the condition that

$$R(t,t') \to \ln(z - z' - \theta\theta') \quad \text{at} \quad z \to z' \quad . \tag{34}$$

In this case

$$J_r(t_s^a) = J_r^{(s)}(t) + 2\pi i\delta_{rs}, \qquad J_r(t_s^b) = J_r(t) + 2\pi i\omega_{sr} \tag{35}$$

where ω is the period matrix. The above $R(t,t')$ is defined up to terms due to scalar zero modes, which do not contribute into superstring amplitudes. Apart from an additive constant the discussed zero mode terms can be fixed by the condition that Green function $K(t,t')$ being defined as

$$K(t,t') = D(t')R(t,t'), \tag{36}$$

decreases to zero when $z \to \infty$ or $z' \to \infty$. In eq.(36) the spinor derivative $D(t')$ is defined by (27). Then, for even superspin structures, eqs. (32)-(35) fully determine the above $K(t,t')$. Then $R(t,t')$ appears to be determined up to an unessential additive constant. It is obvious also that the discussed $K(t,t')$ are 1/2-supertensors on t'-supermanifold. But on t-supermanifold the above $K(t,t')$ functions have the periods $2\pi i\eta_s(t')$ where 1/2-superdifferentials $\eta_s(t')$ are defined as

$$2\pi i\eta_s(t') = D(t')J_s(t'). \tag{37}$$

The scheme discussed in Sec.II for the boson string can be extended [11,12] to the superstring theory, as well. In this case the equations for the partition functions in (15) are derived from the condition that the multi-loop amplitudes are independent of a choice of both the *vierbein* and the gravitino field. In details the proof of the above equations has been given in [11,12], the final results being given in this paper just below.

In the superstring theory there are 3/2-tensor ghost superfield $\hat{B}$ and the vector ghost one $\hat{F}$. In the considered scheme [11,12] the above vector superfield has depending on t periods under rounds about (A_s, B_s)-cycles. For this reason the vacuum correlator

$$G_{gh}(t,t') = < \hat{F}(t,\bar{t})\hat{B}(t',\bar{t}') > \tag{38}$$

also has depending on t periods under rounds about (A_s, B_s)-cycles. In the explicit form

$$G_{gh}(t_s^a,t') = Q_{\Gamma_{a,s}}^{-2}(t)\left(G_{gh}^{(s)}(t,t') + \sum_N Y_{a,N}^{(s)}(t)\tilde{\chi}_N(t')\right) \quad ,$$

$$G_{gh}(t_s^b,t') = Q_{\Gamma_{b,s}}^{-2}(t)\left(G_{gh}(t,t') + \sum_N Y_{b,N}^{(s)}(t)\tilde{\chi}_N(t')\right) \tag{39}$$

where $t_s^a = (g_s^a|\theta_s^a)$ and $t_s^b = (g_s^b|\theta_s^b)$ are the same as in (29) and $\tilde{\chi}_N$ are 3/2-zero modes. And $G_{gh}^{(s)}(t,t')$ has been obtained from $G_{gh}(t,t')$ by 2π-twist about $\hat{C}_s^{(-)}$-circle (20). Furthermore, $Y_{a,N}^{(s)}$ and $Y_{b,N}^{(s)}$ are polynomials [12] of degree 2 in (z,θ), which are given by

$$Y_{p,N_r}^{(s)}(t) = Y_{p,N_s}(t)\delta_{rs} \quad \text{where} \quad p = a,b \quad \text{and} \quad Y_{p,N_s}(t) = Q_{\Gamma_{p,s}}^{2}\left[\frac{\partial g_s^p}{\partial q_{N_s}} + \theta_s^p\frac{\partial\theta_s^p}{\partial q_{N_s}}\right]. \tag{40}$$

As it was explained just after eq.(19), among $(3n|2n)$ Schottky parameters there are the $\{q_{N_o}\}$ set of $(3|2)$ parameters, which are the same for all the genus-n supermanifolds, and, therefore, they are not moduli. The sum over N in (39) includes only those $Y^{(s)}_{p,N_r}(t)$, which associated with the Schottky parameters that are moduli. But eq.(40) allows to assign the $Y^{(s)}_{p,N_r}(t)$ polynomial to every the Schottly parameter including $\{q_{N_o}\}$, too. As far as $t^a_s = t$ for $l_{1s} = 0$, the $Y_{a,N_s}(t)$ polynomials are unequal to zero only, if $l_{1s} \neq 0$. In this case, for every N_s, the $Y_{a,N_s}(t)$ function is equal to $Y_{b,N_s}(t)$ calculated at $\sqrt{k_s} = -1$. The last statement follows from eq.(24). If $\mu_s = \nu_s = 0$, then among the above $Y_{a,N_s}(t)$ functions only $Y_{a,\mu_s}(t)$ and $Y_{a,\nu_s}(t)$ are unequal to zero.[3] And $Y_{a,k_s}(t) = 0$ in any case. For every s, the $Y_{b,N_s}(t)$ functions form the full set of the degree-2 polynomials in (z,θ). In the terms of Schottky parameters the $Y_{b,N_s}(t)$ functions have been obtained in [12], see also eqs. (A1) and (A2) in Appendix A of the present paper.

Eqs.(39) present the extension of eq.(10) to the superstring theory. The above equations together with the condition that $G_{gh}(t,t')$ is 3/2-superconformal form on t'-supermanifold fully determine [11,12] both $G_{gh}(t,t')$ and 3/2-superconformal zero modes $\tilde{\chi}_N(t')$. Unlike the ghost correlators discussed in [13,14], $G_{gh}(t,t')$ satisfying (10), has no unphisical poles [11,12].

The equations for the partition functions in (15) have been found to be [11,12]

$$\sum_N \tilde{\chi}_N(t)\frac{\partial}{\partial q_N}\ln \hat{Z}^{(n)}_{L,L'} =< T_{gh} + T_m > - \sum_N \frac{\partial}{\partial q_N}\tilde{\chi}_N(t) \tag{41}$$

together with the equations to be complex conjugated to (41), the 3/2- zero modes $\tilde{\chi}_N(t)$ being the same as in (39). The derivatives with respect to odd moduli in eq.(41) are implied to be the "right" ones. The summation in (41) is performed over all the sets $\{N_r\}$ of the $(k_r, u_r, v_r, \mu_r, \nu_r)$ indices except the $\{N_o\}$ set associated with those Schottky parameters that are not the moduli. Furthermore, T_{gh} and T_m are the stress tensors of the ghost and string superfields, respectively. In the explicit form

$$T_m = 10(\partial X)DX/2 \tag{42}$$

where X denotes the scalar superfield, they being in number 10 (10 is the space-time dimension). Furthermore,

$$T_{gh} = -(\partial F)B - \partial(FB) + D[(DF)B]/2 \quad . \tag{43}$$

In eqs. (42) and (43) the explicit dependence on the supercoordinate $t = (z|\theta)$ is omitted. Being written for all superspin structures, eqs.(41) form the set of the equation that is invariant under the supermodular group. The proof of this statement is similar to that in the boson string theory. We plan to give this proof in an another paper.

The above T_m and T_{gh} are calculated in the terms of the vacuum correlators (31) and (38), the singular at $z \to z'$ term being removed, as it was explained in [25]. For the superspin structure $L_0 = \bigcup_s(l_{1s} = 0, l_{2s} = 1/2)$ all the above correlators has been calculated in [11,12]. All they are obtained by the "naive" supersymmetrization of the boson string ones[10]. For the superspin structures $L(0,l_2) = \bigcup_s(l_{1s} = 0, l_{2s})$ all the discussed correlators are obtained from those for the L_0 superspin structure by the $\sqrt{k_s} \to -\sqrt{k_s}$ replacement for every $\sqrt{k_s}$ associated

[3] Throughout this paper we use for the $\{N_r\}$ indices the same notation $(k_r, u_r, v_r, \mu_r, \nu_r)$ as for the Schottky parameters. In this notation, particularly, $q_{k_r} = k_r$, $q_{u_r} = u_r$ and so one.

with $l_{2s} = 0$. If at least one $l_{1s} \neq 0$ and odd moduli are arbitrary, a fermion-boson mixing arises under twists about both A- and B-cycles that prevents to construct superfield vacuum correlators in the form of Poincaré series [14]. The method calculating the discussed correlators is given below, the even superspin structures being considered.

4 Scalar Supermultiplets

As it has been noted in the previous Section, the vacuum correlators (31) of the scalar superfields is expressed in the terms of the holomorphic Green functions $R(t,t')$. If all the l_{1s} characteristics are equal to zero, the above $R(t,t')$ have been obtained [11,12,22,23,30] by a simple extension of the boson string result [14,25]. In this Section we calculate the discussed $R(t,t')$ for those even superspin structures where at least one of the above l_{1s} characteristics is unequal to zero.

Even for the discussed superspin structures, there are no special difficulties in the calculation of $R(t,t')$, if all the odd parameters are equal to zero. In this case $R(t,t')$ is reduced to Green function $R_{(o)}(t,t')$ that is written in the terms of both the boson Green function $R_b(z,z')$ and the fermion Green one $R_f(z,z')$ as

$$R_{(o)}(t,t') = R_b(z,z') - \theta\theta' R_f(z,z'). \tag{44}$$

The boson Green function, as well as its periods $J_{(o)s}$ together with the period matrix $\omega^{(o)}$ have been calculated in [14] (see also Appendix B of the present paper). The fermion Green function $R_f(z,z')$ in (44) has been discussed in [18]. We write $R_f(z,z')$ in the following form:

$$R_f(z,z') = \exp\left\{\frac{1}{2}[R_b(z,z) + R_b(z',z')] - R_b(z,z')\right\} \frac{\Theta[l_1,l_2](J|\omega)}{\Theta[l_1,l_2](0|\omega)} \tag{45}$$

where Green function $R_b(z,z)$ for $z'=z$ is defined to be the limit of $R_b(z,z') - \ln(z-z')$ at $z \to z'$. Furthermore, Θ is the theta function and the symbol J denotes the set of functions $(J_{(o)s}(z) - J_{(o)s}(z'))/2\pi i$.

Odd moduli being arbitrary, it is not a simple deal to satisfy eqs.(32) because in this case fermions are agitated with bosons under rounds above both B_s and A_s cycles. To solve this problem we built, for every s, the genus-1 Green function $R_s^{(1)}(t,t')$ that being twisted under (A_s,B_s)-circles, is changed by $(\Gamma_{a,s},\Gamma_{b,s})$-mappings (22). For odd Schottky parameters (μ_s,ν_s) to be equal to zero, $R_s^{(1)}(t,t')$ is reduced to $R_{(o)s}^{(1)}(t,t')$. Using eq.(44), one can calculate this $R_{(o)s}^{(1)}(t,t')$ in the terms of both the genus-1 boson Green function $R_{(b)s}^{(1)}(z,z')$ and the genus-1 fermion Green function $R_{(f)s}^{(1)}(z,z';l_{1s},l_{2s})$. The odd parameters being arbitrary, $R_s^{(1)}(t,t')$ has the following form:

$$R_s^{(1)}(t,t') = R_{(o)s}^{(1)}(t_s,t'_s) + \tilde{\varepsilon}'_s\theta'_s\Xi_s(\infty,z'_s) - \theta_s\tilde{\varepsilon}'_s\Xi_s(z_s,\infty) \quad \text{for} \quad s=1,2,...n \tag{46}$$

where both $t_s = (z_s|\theta_s)$, $t'_s = (z'_s|\theta'_s)$ and $\tilde{\varepsilon}'_s$ are defined by (23). Furthermore, $\Xi_s(z,z')$ is

$$\Xi_s(z,z') = (z-z')R_{(f)s}^{(1)}(z,z';l_{1s},l_{2s}) \tag{47}$$

Apart from two the last terms, $R_s^{(1)}(t,t')$ is obtained by the $\tilde\Gamma_s$ transformation of $R_{(o)s}^{(1)}(t,t')$ that corresponds to eq.(22). Two the last terms in (46) are proportional to scalar zero mode on the t or t' supermanifold. They are introduced in (46) to provide decreasing $K_s^{(1)}(t,t')$ at $z\to\infty$ or $z'\to\infty$ where $K_s^{(1)}(t,t')$ is defined by eq.(36) for $R=R_s^{(1)}$.

To calculate $R_{(o)s}^{(1)}(t,t')$ for even genus-1 spin structures we use eq.(45) at $n=1$. But among even genus-n superspin structures one may see an even number of the handles associated with the odd genus-1 superspin structure ($l_{1s}=l_{2s}=1/2$). In this case, because the genus-1 fermion zero mode presents, the genus-1 fermion Green functions $R_{(f)s}^{(1)}(z,z';1/2,1/2)$, generally, have periods under twists about both A_s and B_s cycles. And there is the Green function among of them, which has the periods only about B_s-cycle. The above Green function is given by

$$R_{(f)s}^{(1)}(z,z')=\frac{\partial_z\{\Theta[1/2,1/2](J_{(1)}|\omega_s^{(1)})\}}{\Theta[1/2,1/2](J_{(1)}|\omega_s^{(1)})}\sqrt{\frac{\partial_{z'}J_{(o)s}^{(1)}(z')}{\partial_z J_{(o)s}^{(1)}(z)}} \tag{48}$$

where Θ is the genus-1 theta function. Furthermore, $J_{(1)}=(J_{(o)s}^{(1)}(z)-J_{(o)s}^{(1)}(z'))/2\pi i$ and $J_{(o)s}^{(1)}$ is the period of $R_{(b)s}^{(1)}(z,z')$, the period of $J_{(o)s}^{(1)}$ being $2\pi i\omega_s^{(1)}$. In this case, for every s, the Green function $K_s^{(1)}(t,t')=D(t')R_s^{(1)}(t,t')$ is changed under $\Gamma_{b,s}$ transformation as

$$\begin{aligned}K_s^{(1)}(t,t_s^b(t'))&=\left[K_s^{(1)}(t,t')+\varphi_s(t)f_s(t')\right]Q_{\Gamma_{b,s}}(t')\\ K_s^{(1)}(t_s^b(t),t')&=K_s^{(1)}(t,t')+2\pi i\eta_s^{(1)}(t')-\varphi_s(t)f_s(t')\end{aligned} \tag{49}$$

where $f_s(t')=D(t')\varphi_s(t')$. The above $\varphi_s(t')$ disappears, if (l_{1s},l_{2s})-characteristics correspond to an even genus-1 spin structure. Besides, one can prove that [4]

$$\int\limits_{C_b^{(s)}} f_s(t)\frac{d\theta dz}{2\pi i}\varphi_s(t)=-1,\quad \int\limits_{C_b^{(s)}} f_s(t)\frac{d\theta dz}{2\pi i}=0\quad\text{and}\quad \int\limits_{C_b^{(s)}} K_s^{(1)}(t,t')\frac{d\theta' dz'}{2\pi i}=0. \tag{50}$$

In (50) integrating over z is performed along the contour $C_b^{(s)}$. On the z_s complex plane (23) this $C_b^{(s)}$ contour is none other than the $\hat C_s^{(-)}$ circle (20), see Appendix C for more details. It is worth- while to note that the last equation of eqs.(50) remains to be true, if one take $K(t,t')$ instead of $K_s^{(1)}(t,t')$. To prove (50) we start with the following relations:

$$f_s(t)=-\int\limits_{C(z)} f_s(t_1)dt_1K_s^{(1)}(t_1,t)\quad\text{and}\quad K_s^{(1)}(t,t')=\int\limits_{C(z)}K_s^{(1)}(t,t_1)dt_1K_s^{(1)}(t_1,t') \tag{51}$$

where $dt_1=d\theta_1 dz_1/2\pi i$. In (51) the infinitesimal contour $C(z)$ gets around z-point in the positive direction. Then we deform this contour to the C_s contour that surrounds both $C_s^{(-)}$ and $C_s^{(+)}$ circles (26) together with the $\tilde C_s$ cut arising for $l_{1s}\neq 0$. In the second of eqs.(51) the additional term arises due to the pole at $z_1=z'$. The above pole contribution is $K_s^{(1)}(t,t')$. Therefore, in this case the integral about the C_s contour is equal to zero. Using eqs.(49), we

[4]Throughout the paper $\int d\theta\theta=1$

reduced the integrals about the C_s contour to those presented in (50). Besides, from (51) it is follows (see Appendix C for more details) that

$$\int\limits_{C_b^{(s)}} K_s^{(1)}(t,t')\frac{d\theta' dz'}{2\pi i}\varphi_s(t') = \frac{\varphi_s(t')}{2}, \quad \int\limits_{C_b^{(s)}} f_s(t)\frac{d\theta dz}{2\pi i}K_s^{(1)}(t,t') = -\frac{f_s(t')}{2} \tag{52}$$

The second of eqs.(52) follows from the first one because $f_s(t) = D(t)\varphi_s(t)$ and $D(t)R_s^{(1)}(t,t') = K_s^{(1)}(t',t)$.

It is much more convenient to have deal with Green function $K(t,t')$ instead of $R(t,t')$ where $K(t,t')$ is defined by (36). The above Green function satisfy the following relation:

$$K(t,t') = K_{(o)}(t,t') - \sum_{r=1}^{n}\int\limits_{C_r} K_{(o)}(t,t_1)dt_1\delta K_r^{(1)}(t_1,t_2)dt_2K(t_2,t') - \int\limits_{C_r} K_{(o)}(t,t_1)\times$$

$$\delta\varphi_r(t_1)dt_1\int\limits_{C_b^{(r)}} f_r(t_2)dt_2K(t_2,t') + \int\limits_{\hat{C}_r^{(-)}} K_{(o)}(t,t_1)\varphi_{(o)r}(t_1)dt_1\int\limits_{C_r}\delta f_r(t_2)dt_2K(t_2,t') \tag{53}$$

where $dt = d\theta dz/2\pi i$, etc. Furthermore, $K_{(o)}(t,t')$ and $\varphi_{(o)r}$ denote, respectively, $K(t,t')$ and φ_r calculated at all odd Schottky parameters to be equal to zero.

In the first integral on the right side of eq.(53) the integrations over both z_1 and z_2 are performed along C_r-contour that gets around in the positive direction both $C_r^{(-)}$ and $C_r^{(+)}$ circles (26) together with the $\tilde{C}_r$ cut, if this cut presents (i.e. $l_{1r} \neq 0$). Furthermore, $C_b^{(r)}$-contour is the same as in (50) and (52), $\hat{C}_r^{(-)}$ being defined by (20).

Each of $\delta K_r^{(1)}, \delta\varphi_r$ and δf_r in (53) is defined to be the difference between the corresponding value and that calculated for zero values of odd Schottky parameters (μ_r, ν_r). As example, $\delta K_r^{(1)}(t_1,t_2) = K_r^{(1)}(t_1,t_2) - K_{(o)r}^{(1)}(t_1,t_2)$ where $K_{(o)r}^{(1)}(t_1,t_2)$ is the $K_r^{(1)}(t_1,t_2)$ function taken at $\mu_r = \nu_r = 0$. So the first integral on the right side could be written down to be the difference of two integrals, the integrands containing $K_r^{(1)}(t_1,t_2)$ and $K_{(o)r}^{(1)}(t_1,t_2)$, respectively. In the first of these integrals the integration over z_2 along C_r-contour could be reduced to the integration performed along the $C_b^{(r)}$ one. The second of these integrals being considered, the integration over z_1 could be reduced to the integration along $\hat{C}_r^{(-)}$-contour. After the above procedure to be made, one can verify that the right side of (53) is equal to $K(t,t')$ that proves eq.(53), for more details see eqs. (C12) and (C13) in Appendix C. Only the odd genus-1 spin structures contribute in two last terms on the right side of (53).

The term $K_{(o)}(t,t') = D(t')R_{(o)}(t,t')$ outside the integral on the right side of (53) is calculated in the terms of both $R_b(z,z')$ and $R_f(z,z')$, as it has been explained above. So, (53) appears to be an integral equation for $K(t,t')$. As far as the kernel of this equation is proportional to odd parameters, the discussed equation has the unique solution. The above solution can be obtained by the iteration procedure, every posterior iteration being, at least, one more power in odd parameters than a previous one. Therefore, $K(t,t')$ appears to be a series containing a finite number of terms. One can verify that the discussed solution possesses the required properties (32)-(35). Details of this verification are planned to give in an another paper.

After $K(t,t')$ being determined, the desirable Green function $R(t,t')$ is calculated without essential difficulties. It is convenient to determine its periods J_r from the following relation:

$$J_s(t) = \int\limits_{C_s} K(t,t') J_s^{(1)}(t') d\theta' dz' \tag{54}$$

where $J_r^{(1)}(t')$ is the period of the genus-1 Green function $R_r^{(1)}(t,t')$, the period of $J_r^{(1)}(t')$ being $\omega_r^{(1)}$. The period matrix ω_{rp} is given by

$$\omega_{rp} = \omega_r^{(1)}\delta_{rp} + \int\limits_{C_r} \eta_p(t) J_r^{(1)}(t) d\theta dz = \omega_r^{(1)}\delta_{rp} + \int\limits_{C_p} \eta_r(t) J_p^{(1)}(t) d\theta dz \tag{55}$$

where $\eta_r(t)$ is defined by (37). In eqs. (54) and (55) the integration contour C_r is the same as in (53), the proof of both (54) and (55) see in Appendix C. If all the l_{1s} characteristics are equal to zero, one can obtain the much more simple formulae [14,22,23] for the discussed Green functions, the J_s functions and the period matrices, as well. But it seems that for the rest of the superspin structures the above values can hardly be obtained in a much more compact form. Though the discussed expressions for the above values seem to be rather complicate, they are quite appropriate for investigating almost degenerated surfaces. Therefore, one can hope, at least, to study the divergency problem for the superstring amplitudes.

In addition to (53), we obtain some another equations that will be convenient in Sec.VI to calculate the partition functions. For this aim we start with the representations of $K(t,t')$ in the form of the integrals along the infinitesimal contour $C(z')$ that is the same as in (51), the integrands being $K_s^{(1)}(t,t_1)K(t_1,t')$ where $s = 1,2,...n$. The above $C(z')$ contour being deformed, the discussed relations are transformed into the following ones:

$$K(t,t') = K_s^{(1)}(t,t') + \sum_{r\neq s}\int\limits_{C_r} K_s^{(1)}(t,t_1)\frac{d\theta_1 dz_1}{2\pi i}K(t_1,t') - \varphi_s(t)\int\limits_{C_b^{(s)}} f_s(t_1)\frac{d\theta_1 dz_1}{2\pi i}K(t_1,t') \tag{56}$$

where C_r-contours are the same as in (53) and $s = 1,2,...n$. There is no the term with $r = s$ in the sum over r because the discussed term is given by the integral along $C_b^{(s)}$- contour on the right side of (56), $C_b^{(s)}$- contour being the same as in (50). Indeed, as far as the $\Gamma_{a,s}$ and $\Gamma_{b,s}$ mappings are the same for both $K_s^{(1)}$ and K, the discussed integral along the C_s contour can be reduced to the one along $C_b^{(s)}$. The difference of the resulting integral and the last term in eq.(56) appears to be

$$-\int\limits_{C_b^{(s)}} \left[K_s^{(1)}(t,t'') + \varphi_s(t) f_s(t'')\right]\frac{d\theta'' dz''}{2\pi i} 2\pi i \eta_s(t') \tag{57}$$

that is equal to zero because of (50) and (52).

Every C_r-contour being considered, we denote K(t,t') for $z \in C_r$ as $\tilde{K}_r(t,t')$. We use (56) to calculate the set $\{\tilde{K}\}$ of the above $\tilde{K}_r$ functions. As soon as the C_r contours can be moved, all the above $\tilde{K}_r$ determine in fact the same function $K(t,t')$. So, a choice of either $\tilde{K}_r$ to fit

the discussed K is only a matter of a convenience. The considered set of equations (56) can be also written down in an operator form as

$$\tilde{K} = \tilde{K}^{(1)} + (\hat{K}^{(1)} - \hat{\varphi}\hat{f})\tilde{K} \tag{58}$$

where $\tilde{K}^{(1)} = \{\tilde{K}_s^{(1)}\}$ denotes the set of the $K_s^{(1)}(t,t')$ functions, every $K_s^{(1)}(t,t')$ being taken at $z \in C_s$. Furthermore, $\hat{\varphi} = \{\hat{\varphi}_{sr}\}$ is the diagonal matrix depending on t. The matrix elements of this matrix are $\hat{\varphi}_{ss} = \varphi_s(t)$ and $\hat{\varphi}_{sr} = 0$ for $s \neq r$. Both matrices $\hat{K}^{(1)} = \{\hat{K}_{sr}^{(1)}\}$ and $\hat{f} = \{\hat{f}_{sr}\}$ are the integral operators. For $s \neq r$, the $\hat{K}_{sr}^{(1)}$ operator performs the integration over t' along C_r-contour, the "kernel" being $K_{sr}^{(1)}(t,t')dt'$. We define the "kernel" together with the differential dt' to have deal with the objects obeying bose statistics. Furthermore, $\hat{K}_{ss}^{(1)} = 0$. By contrast, the kernel of the $\hat{f}_{sr}$ operator is unequal to zero only for $s = r$. And the $\hat{f}_{ss}$ operator performs the integration over t' along $C_b^{(s)}$-contour, its kernel being $f_s(t')dt'$. So, being applied to $\tilde{K}_r$, the $\hat{K}_{sr}^{(1)}$ operator reproduces the integral over C_r in (56). Moreover, $\hat{\varphi}\hat{f}\tilde{K}$ gives the integral over $C_b^{(s)}$ on the right side of (56). The first term on the right side of (56) corresponds to the first term on the right side of (58).

For both $z \in C_r$ and z' to be situated inside C_s-contour (but outside $C_b^{(s)}$-contour) the solution of (58) can be written down as

$$\begin{aligned} K_{rs} = K_{rs}^{(1)}\delta_{rs} + \hat{K}_{rs}K_{ss}^{(1)} + \frac{1}{2}\varphi_r V_{rs}^{-1} f_s - \sum_m \varphi_r V_{rm}^{-1}\hat{f}_{mm}\hat{K}_{ms}K_{ss}^{(1)} + \\ \frac{1}{2}\sum_n \hat{K}_{rn}\hat{\varphi}_{nn}V_{ns}^{-1}f_s - \sum_{q,m}\hat{K}_{rq}\hat{\varphi}_{qq}V_{qm}^{-1}\hat{f}_{mm}\hat{K}_{ms}K_{ss}^{(1)} \end{aligned} \tag{59}$$

where K_{rs} and $K_{rs}^{(1)}$ specifies $K(t,t')$ and $K^{(1)}(t,t')$, respectively, provided that $z \in C_r$, z' being situated inside C_s (but outside $C_b^{(s)}$-contour). Moreover, $f_s = f_s(t')$ and $\varphi_r = \varphi_r(t)$ where $\varphi_r(t)$ and $f_s(t')$ are defined by (49). Both integral operator $\hat{K} = \{\hat{K}_{pq}\}$ and the matrix $V = \{V_{mn}\}$ are defined as

$$\hat{K} = (I - \hat{K}^{(1)})^{-1} - I \quad \text{and} \quad V_{mn} = \sum_p \hat{f}_{mp}\hat{K}_{pn}\varphi_n \tag{60}$$

where I is the identical operator and $\hat{K}^{(1)}$ is the same as in (58). The $\{V_{mn}\}$ matrix is defined only for those (m,n), which label the odd genus-1 spin structures. So, for superspin structures where no the genus-1 odd superspin ones only two first terms remain on the right side of(59). In this case

$$K_{rs} = K_{rs}^{(1)}\delta_{rs} + \hat{K}_{rs}K_{ss}^{(1)}. \tag{61}$$

Eqs. (59) and (61) can be verified by the substitution into eq.(58). As for eq.(53) it can be proved that the solution of (56) possesses the required properties (32)-(35), but in details this matter is planned to discuss in an another paper. Eqs. (59) and (61) will be used in Sec.VI for the calculation of the partition functions.

5 Ghost Vacuum Correlators

As it has been explained in Sec.III, the ghost vacuum correlators $G_{gh}(t,t')$ satisfy eqs.(39). For the superspin structures where $l_{1s}=0$ for every s, the above correlators have been obtained in [11,12,30]. In this Section we calculate the discussed correlators for those even superspin structures where at least one of l_{1s}-characteristics is unequal to zero.

It is convenient to write [11,12] the discussed $G_{gh}(t,t')$ in the terms of a new Green function G(t,t') as

$$G_{gh}(t,t')=G(t,t')-\sum_{N_o,N'_o}Y_{b,N_o}(t)A^{-1}_{N_oN'_o}\chi_{N'_o}(t') \qquad (62)$$

where χ_{N_o} are 3/2-supertensors, A_{N_o,N'_o} is a matrix that will be defined below and $Y^{(s)}_{b,N_o}(t)$ are defined by (40) at $N_r=N_o$. In contrast to eqs.(39), the summation in (62) is performed over those the N_o indices that are associated with those (3|2) Schottky parameters q_{N_o}, which are not moduli. The $G(t,t')$ Green function changes under the $\Gamma_{a,s}(l_{1s})$ and $\Gamma_{b,s}(l_{2s})$) mappings as follows

$$G(t^a_s,t')=Q^{-2}_{\Gamma_{a,s}}(t)\left(G^{(s)}(t,t')+\sum_{r=1}^{n}\sum_{N_r}Y^{(s)}_{a,N_r}(t)\chi_{N_r}(t')\right)\quad,$$

$$G(t^b_s,t')=Q^{-2}_{\Gamma_{b,s}}(t)\left(G(t,t')+\sum_{r=1}^{n}\sum_{N_r}Y^{(s)}_{b,N_r}(t)\chi_{N_s}(t')\right) \qquad (63)$$

where $G^{(s)}(t,t')$ has been obtained from $G(t,t')$ by 2π-twist about $\hat{C}^{(-)}_s$-circle (20). Unlike (39), the summation in (63) performs over all the N_r indices, the $\{N_o\}$ set being among of them. The polynomials $Y^{(s)}_{a,N_r}(t)$ and $Y^{(s)}_{b,N_r}(t)$ are defined for all N_r by (40). As it follows from (40), only the $r=s$ terms contribute to (63).

The A_{N_o,N'_o} matrix in (62) is the square submatrix $B_{N_oN'_o}$ of the $B_{N_sN_r}$ matrix that is determined by the following relations:

$$Y_{b,N_r}(t^p_s)=Q^{-2}_{\Gamma_{p,s}}(t)\left(Y_{b,N_r}(t)+\sum_{N_s}Y_{p,N_s}(t)B_{N_sN_r}\right). \qquad (64)$$

One can calculate the discussed $B_{N_sN_r}$ matrix using eqs. (A1) together with (A3) and (A4). It is quite important that $B_{N_sN_r}$ appear to be the same for both $t\to t^a_s$ and $t\to t^b_s$ mappings. So $G(t,t')$ being defined by(62), satisfies both equations (39). It is obvious that the discussed $B_{N_sN_r}$ are independent of the superspin structure. Furthermore, one can verify that $B_{N_rk_r}=B_{k_rN_r}=0$. Also, it can be noted that the $\tilde{\chi}_N$ zero modes are calculated in the terms of χ_N as

$$\tilde{\chi}_{N_s}(t)=\chi_{N_s}(t)-\sum_{N_o,N'_o}B_{N_sN_o}A^{-1}_{N_oN'_o}\chi_{N'_o}(t'). \qquad (65)$$

Under the $\Gamma_{a,s}(l_{1s})$ and $\Gamma_{b,s}(l_{2s})$ mappings on the t' supermanifold the discussed $G(t,t')$ is due to be 3/2-superconformal tensor. This condition together with eqs.(63) determines in

the unique way both $G(t,t')$ and χ_N. To prove this statement we are based on the following relations:

$$\int\limits_{C_b^{(s)}} G(t,t')\frac{d\theta' dz'}{2\pi i}Y_{b,N_s}(t') + \int\limits_{C_a^{(s)}} G(t,t')\frac{d\theta' dz'}{2\pi i}Y_{a,N_s}(t') = 0 \tag{66}$$

where the $C_b^{(s)}$ contour is the same as in (50) and (C10). The integral along $C_a^{(s)}$ presents in (66) only for $l_{1s} \neq 0$. As it has been noted in Sec.III, in this case the cut $\tilde{C}_s$ appears on the considered z-plane. The one of its endcut points is placed inside the $C_s^{(-)}$ circle and the other endcut point is placed inside the $C_s^{(+)}$ one, both the circles being defined by (20). The $C_b^{(s)}$ contour being considered on the z_s complex plane (23), starts with the point $z_s^{(-)}$ of the intersect of the $\hat{C}_s^{(-)}$ contour with the above $\tilde{C}_s$ cut. On the considered z_s complex plane (23) the $C_a^{(s)}$ path goes along the low edge of the $\tilde{C}_s$ cut from the $z_s^{(-)}$ point to the intersect $z_s^{(+)} = \tilde{C}_s \bigcap \hat{C}_s^{(+)}$ that is chosen to be the same as in (C10) of Appendix C. Eqs.(66) are proved in the manner that is quite similar to the proof of eqs.(50), for more details see eq.(C14) of Appendix C.

To prove that $G(t,t')$ is determined by (63) in the unique way, we assume that there are two different Green functions $G_1(t,t')$ and $G_2(t,t')$ satisfying (63) and consider the difference $\delta G(t,t') = G_1(t,t') - G_2(t,t')$. The above $\delta G(t,t')$ has not the pole at $z = z'$. Besides, it satisfies eqs.(63) with some 3/2-supertensors $\delta\chi_{N_s}(t')$ instead of $\chi_{N_s}(t')$. Furthermore, δG is represented by the integral over t'' performed along infinitesimal contour $C(z)$, which surrounds z-point, the integrand being $\delta G(t,t'')G_1(t'',t')$. Being reduced to the integrals along both $C_b^{(s)}$ and $C_b^{(s)}$ contours, the above integral is found to be equal to zero owing to (66). Therefore, $\delta G(t,t') \equiv 0$. It proves the desired statement that the $G(t,t')$ Green function is unique. At the same time eqs.(63) determine $\chi_{N_s}(t')$ in the unique way, too. A like consideration could be performed also to prove the uniqueness of $G_{gh}(t,t')$.

If all the odd parameters are equal to zero, $G(t,t')$ is reduced to $G_{(o)}(t,t')$ that is given in the terms of both the boson Green function $G_b(z,z')$ and the fermion Green function $G_f(t,t')$ as

$$G_{(o)}(t,t') = G_b(z,z')\theta' + \theta G_f(z,z'). \tag{67}$$

The boson Green function $G_b(z,z')$ has been given in [10] as

$$G_b(z,z') = -\sum_{\Gamma} \frac{1}{[z - g_\Gamma(z')][c_\Gamma z' + d_\Gamma]^4}. \tag{68}$$

In (68) the summation is performed over all the group products $\Gamma = \{z \to g_\Gamma(z) = (a_\Gamma z + b_\Gamma)(c_\Gamma z + d_\Gamma)^{-1}\}$ of the basic Schottky group elements $\Gamma_s = \{z \to g_s(z)\}$. As it will be explained below, the fermion Green function $G_f(z,z')$ in eq.(67) is calculated in the terms of the following $G_{(\sigma)}(z,z')$ Green functions:

$$G_{(\sigma)}(z,z') = \sum_{\Gamma} \frac{\exp \pi i[\Omega_\Gamma(\{\sigma_s\}) + \sum_s 2l_{1s}\sigma_s(J_{(o)s}(z) - J_{(o)s}(z'))]}{[z - g_\Gamma(z')][c_\Gamma z' + d_\Gamma]^3} \tag{69}$$

where $\sigma_s = \pm 1$. So, $G_{(\sigma)}$ depends on a choice of the $\{\sigma_s\}$ set. The value $\Omega_\Gamma(\{\sigma_s\})$ in (69) is defined as

$$\Omega_\Gamma(\{\sigma_s\}) = -\sum_{s,r} 2l_{1s}\sigma_s\omega_{sr}^{(o)} n_r(\Gamma) + \sum_r (2l_{2r} - 1)n_r(\Gamma) \tag{70}$$

where $n_r(\Gamma)$ is the number of times that the Γ_r generators are present in Γ (for its inverse $n_r(\Gamma)$ is defined to be negative). Furthermore, $J_{(o)s}$ in (69) are the periods of the boson Green function $R_b(z,z')$ in eq.(44), and $\omega_{sr}^{(o)}$ in (70) is the period matrix at zero odd moduli.

For zero values of the odd Schottky parameters, 2π-twist about every B_s-cycle is given by the mapping $\{z \to z_{(o)r} = g_r(z), \theta \to \theta_{(o)r} = -(-1)^{l_{1r}+l_{2r}}(c_r z + d_r)^{-1}\theta)\}$. It is follows from (69) that the changes of $\theta G_{(\sigma)}$ under the above mappings are

$$\theta_{(o)r} G_{(\sigma)}(z_{(o)r}, z') = (c_r z + d_r)^{-2} \left(\theta G_{(\sigma)}(z,z') + \sum_{N_r=\mu_r,\nu_r} \tilde{Y}_{\sigma,N_r}(t) \Phi^{(0)}_{\sigma,N_r}(z') \right) \tag{71}$$

where $\Phi^{(0)}_{\sigma,N_r}(t')$ are 3/2-tensors and $\tilde{Y}_{\sigma,N_r}(t)$ for $N_r = (\mu_r, \nu_r)$ is given by

$$\tilde{Y}_{\sigma,N_r}(t) = \exp[\pi i \textstyle\sum_s 2 l_{1s} \sigma_s J_{(o)s}(z)] Y^{(0)}_{b,N_r}(t) \tag{72}$$

where $Y^{(0)}_{b,N_r}(t)$ is equal to $Y_{b,N_r}(t)$ in (40) at $\mu_r = \nu_r = 0$. One can verify that the above conditions (71) differ from those for $G_f(z,z')$ in eq.(68). The conditions for $G_f(z,z')$ are derived from eqs.(63) at zero odd Schottky parameters.

To obtain the $G_f(z,z')$ Green function we write $G_f(z,z')$ as the integral over t'' performed along the infinitesimal contour $C(z')$ around z', the integrand being $G_{(\sigma)}(z,z'')\theta'' G_f(z'',z')$. Running this contour away, we obtain the following expression for $G_f(z,z')$ in question:

$$G_f(z,z') = G_{(\sigma)}(z,z') - \sum_{s=1}^{n} \sum_{p=a,b} \sum_{N_s} \int\limits_{C_p^{(s)}} G_{(\sigma)}(z,z'') \frac{d\theta'' dz''}{2\pi i} Y^{(0)}_{p,N_s}(t'') \chi^{(0)}_{N_s}(z') \tag{73}$$

where $N_s = (\mu_s, \nu_s)$ and $\chi^{(0)}_{N_s}(z')$ are 3/2-tensors. The $C_p^{(s)}$-contours (where $p = a, b$) are the same as in (66). Furthermore, $Y^{(0)}_{p,N_s}$ are defined to be Y_{p,N_s} in eq.(40) at $\mu_r = \nu_r = 0$. Eq.(73) follows from eq.(C14) in Appendix C. As soon as there is the only $G_{(o)}(z,z')$ Green function, $G_f(z,z')$ determined by eq.(73) is, in fact, independent of $\{\sigma_s\}$. The above $\chi^{(0)}_{N_s}(z')$ in (73) are calculated in the terms of $\Phi^{(0)}_{\sigma,N_s}$ in (71) from the following relations:

$$\Phi^{(0)}_{\sigma,N_s} = \sum_{N_r=\mu_r,\nu_r} \tilde{M}_{N_s,N_r}(\{\sigma_q\}) \chi^{(0)}_{N_r} \quad \text{where} \quad \tilde{M}_{N_s,N_r}(\{\sigma_q\}) = \sum_{p=a,b} \int\limits_{C_p^{(r)}} \Phi^{(0)}_{\sigma,N_s}(z) \frac{d\theta dz}{2\pi i} Y^{(0)}_{p,N_r}(t). \tag{74}$$

Eqs.(74) are obtained from condition that, being calculated from (73), the changes under 2π-twists about A_s, B_s-cycles of $G_f(z,z')$ are given by (63) at zero values of the odd Schottky parameters. So, the desired $G_{(o)}(t,t')$ Green function is determined by (68), (73) and (74)

In the presence of the odd Schottky parameters we calculate $G(t,t')$ in the form of series over the above odd parameters, the zero power term being $G_{(o)}(t,t')$. For this aim we derive the equations similar to (56) for $K(t,t')$. For the superspin structures without the genus-1 odd superspin ones the above equations could be obtained directly for the desired $G(t,t')$ in the terms of $G_s^{(1)}(t,t')$ presenting the $G(t,t')$ Green functions associated with the genus-1

supermanifolds. But for the odd genus-1 (super)spin structures the above $G_s^{(1)}(t,t')$ do not exist. In this case we can use ancillary Green functions $S_\sigma(t,t')$ defined below. In Sec.VI we shall see that these ancillary Green functions appears to be useful for calculation also those superspin structures that do not contain the odd genus-1 superspin ones.

The above genus-n Green functions $S_\sigma(t,t')$ satisfy the following conditions:

$$S_\sigma(t_s^b,t') = Q_{\Gamma_{b,s}}^{-2}(t)\left(S_\sigma(t,t') + \sum_{N_s}\hat{Y}_{\sigma,N_s}^{(1)}(t)\Psi_{\sigma,N_s}(t')\right) \tag{75}$$

where $N_s = (k_s,u_s,v_s,\mu_s,\nu_s)$ and $\Psi_{\sigma,N_s}(t')$ are 3/2-supertensors. The $\hat{Y}_{b,N_s}^{(1)}(t)$ polynomials for $N_s = (k_s,u_s,v_s)$ are equal to $Y_{b,N_s}^{(0)}(t_s)Q_{\tilde{\Gamma}_s}^{-2}(t_s)$ where $Y_{b,N_s}^{(0)}(t)$ are defined by eq.(40) at $\mu_s=\nu_s=0$. For $N_s=(\mu_s,\nu_s)$, the discussed $\hat{Y}_{\sigma,N_s}(t)$ polynomials are equal to $\tilde{Y}_{p,N_r}(t_s)Q_{\tilde{\Gamma}_s}^{-2}(t_s)$, $\tilde{Y}_{\sigma,N_r}(t)$ being defined by (72) for the genus $n=1$. The t_s transformation is defined by (23). In addition to (75), the $S_\sigma(t,t')$ Green function is defined to have no periods about A_s cycles.

The desired $G(t,t')$ in eq.(62) is calculated in the terms of $S_\sigma(t,t')$ as follows

$$G(t,t') = S_\sigma(t,t') - \sum_{s=1}^{n}\sum_{p=a,b}\sum_{N_s}\int\limits_{C_p^{(s)}} S_\sigma(t,t_1)\frac{d\theta_1 dz_1}{2\pi i}Y_{p,N_s}(t_1)\chi_{N_s}(t') \tag{76}$$

where $\chi_{N_s}(t')$ are determine by

$$\Psi_{\sigma,N_s} = \sum_{N_r}M_{N_s,N_r}(\{\sigma_q\})\chi_{N_r} \quad \text{where} \quad M_{N_s,N_r}(\{\sigma_q\}) = \sum_{p=a,b}\int\limits_{C_p^{(r)}}\Psi_{\sigma,N_s}(z)\frac{d\theta dz}{2\pi i}Y_{p,N_r}(t). \tag{77}$$

Eq.(76) is obtained by the same method that has been used to derive eq.(73). Eqs.(77) are obtained from condition that the changes of $G(t,t')$ under 2π-twists about A_s,B_s-cycles are given by (63). Being unique, as it has been proved above, $G(t,t')$ determined by eq.(73) is, in fact, independent of $\{\sigma_s\}$.

If all the odd Schottky parameters are equal to zero, then $S_\sigma(t,t')$ is reduced to $S_{(o)\sigma}$. The above $S_{(o)\sigma}$ has the following form:

$$S_{(o)\sigma}(t,t') = G_b(z,z')\theta' + \theta S_{(f)\sigma}(z,z') \tag{78}$$

where $G_b(z,z')$ is given by (68) and $S_{(f)\sigma}(z,z')$ are calculated in the terms of the above $G_{(\sigma)}(z,z')$ Green function (69), as

$$S_{(f)\sigma}(z,z') = G_{(\sigma)}(z,z') - \sum_{r=1}^{n}\sum_{N_r=\mu_r,\nu_r}\int\limits_{C_b^{(r)}} G_{(\sigma)}(z,z'')\frac{d\theta'' dz''}{2\pi i}\tilde{Y}_{\sigma,N_r}^{(1)}(t'')\Psi_{\sigma,N_r}^{(0)}(z') \tag{79}$$

where $\tilde{Y}_{\sigma,N_r}^{(1)}(t'')$ is equal to $\tilde{Y}_{\sigma,N_r}(t'')$ defined by eq.(72) at the genus $n=1$, and $\Psi_{N_r}^{(0)}(z')$ is equal to $\Psi_{N_s}(t')$ in eq.(75) at zero values of the odd Schottky parameters. The above $\Psi_{\sigma,N_r}^{(0)}(z')$

calculated in the terms of $\Phi^{(0)}_{\sigma,N_s}$ in (71) from the following relations:

$$\Phi^{(0)}_{\sigma,N_s}=\sum_{N_r=\mu_r,\nu_r}\hat{M}_{N_s,N_R}(\{\sigma_p\})\Psi^{(0)}_{\sigma,N_r}\quad\text{where}\quad \hat{M}_{N_s,N_R}(\{\sigma_p\})=\int\limits_{C_b^{(r)}}\Phi^{(0)}_{\sigma,N_s}(z)\frac{d\theta dz}{2\pi i}\tilde{Y}^{(1)}_{\sigma,N_r}(t). \quad (80)$$

Eqs.(80) are kindred eqs.(74). They are obtained from condition that the changes under 2π-twists about A_s, B_s-cycles of the $S_{(f)\sigma}(z,z')$ Green functions in (79) are given by (75) at zero values of the odd Schottky parameters.

At arbitrary odd Schottky parameters the $S_\sigma(t,t')$ Green function can be calculated the equations kindred to eqs.(56) of the previous section. These equations give $S_\sigma(t,t')$ in the form of series over odd parameters, the zero power term being $S_{(o)\sigma}$. To obtain the above equations we use the genus-1 Green functions $S^{(1)}_{\sigma,s}(t,t')$ defined by

$$S^{(1)}_{\sigma,s}(t,t')=Q_{\tilde{\Gamma}_s}(t_s)^{-2}\left[G^{(1)}_b(z_s,z'_s)\theta'_s+\theta_s G^{(1)}_{(\sigma)}(z_s,z'_s)-\tilde{\varepsilon}'_s\Sigma_\sigma(z'_s)\right]Q_{\tilde{\Gamma}_s}(t'_s)^3 \quad (81)$$

where t_s is defined by (23) and the $Q_{\tilde{\Gamma}_s}$ factor is defined by (28). Apart the last term, $S^{(1)}_{\sigma,s}(t,t')$ is obtained by the $\tilde{\Gamma}_s$ transformation (23) of $G^{(1)}_b(z,z')\theta'+\theta G^{(1)}_{(\sigma)}(z,z')$. The last term being 3/2-supertensor, is determined from the condition that $S^{(1)}_{\sigma,s}(t,t')$ decreases at $z\to\infty$ or $z'\to\infty$. It is follows from (81) that $S^{(1)}_{\sigma,s}(t,t')$ has no periods about A_s cycles. And under the $\Gamma_{b,s}$ transformation $\{t\to t^b_s\}$ the discussed $S^{(1)}_{\sigma,s}(t,t')$ is changed as

$$S^{(1)}_{\sigma,s}(t^b_s,t')=Q^{-2}_{\Gamma_{b,s}}(t)\left(S^{(1)}_{\sigma,s}(t,t')+\sum_{N_s}\hat{Y}^{(1)}_{\sigma,N_s}(t)\Psi^{(1)}_{\sigma,N_s}(t')\right) \quad (82)$$

where $N_s=(k_s,u_s,v_s,\mu_s,\nu_s)$. Furthermore, $\hat{Y}^{(1)}_{b,N_s}(t)$ are the same as in (75) and $\Psi^{(1)}_{N_s}(t')$ are 3/2-supertensors.

The desired set of equations for $S_\sigma(t,t')$ is given by

$$S_\sigma(t,t')=S^{(1)}_{\sigma,s}(t,t')+\sum_{r\neq s}\int\limits_{C_r}S^{(1)}_{\sigma,s}(t,t_{(1)})\frac{d\theta_{(1)}dz_{(1)}}{2\pi i}S_\sigma(t_1,t')\quad\text{for}\quad s=1,2,...n \quad (83)$$

Eqs.(83) are obtained by the method employed above to derive eqs.(56). There is no the term with $r=s$ in the sum over r. Indeed, as far as the $\Gamma_{a,s}$ and $\Gamma_{b,s}$ mappings are the same for both S_σ and $S^{(1)}_{\sigma,s}$, the discussed term appears to be equal to zero owing to both (75) and (C15) taken for the genus $n=1$.

Every C_r-contour being considered, we denote $S_\sigma(t,t')$ for $z\in C_r$ as $\tilde{S}_{\sigma,r}(t,t')$. We use (83) to calculate the set $\{\tilde{S}_\sigma\}$ of the above $\tilde{S}_{\sigma,r}$ functions. This procedure is quite similar to that discussed in the previous Section. The discussed set of equations can be written down in the operator form similar to (58) as

$$\tilde{S}_\sigma=\tilde{S}^{(1)}_\sigma+\hat{S}^{(1)}_\sigma\tilde{S}_\sigma \quad (84)$$

where $\tilde{S}^{(1)}_\sigma$ is the set of the $S^{(1)}_{\sigma,s}(t,t')$ functions, matrices $\hat{S}^{(1)}_\sigma=\{\hat{S}^{(1)}_{\sigma,sr}\}$ being the integral operators. For $s\neq r$, the $\hat{S}^{(1)}_{\sigma,sr}$ operator performs the integration over t' along C_r-contour, the kernel being $S^{(1)}_{\sigma,sr}(t,t')$. And $\hat{S}^{(1)}_{\sigma,ss}=0$.

For both $z \in C_r$ and z' to be situated inside C_s-contour, the solution of (84) can be written down as

$$S_{\sigma,rs} = S^{(1)}_{\sigma,rs}\delta_{rs} + \sum_n \hat{S}^{(1)}_{\sigma,rn}(I - \hat{S}^{(1)}_\sigma)^{-1}_{ns} S^{(1)}_{\sigma,ss} \tag{85}$$

where $S_{\sigma,rs}$ and $S^{(1)}_{\sigma,rs}$ specifies $S_\sigma(t,t')$ and $S^{(1)}_\sigma(t,t')$, respectively, provided that $z \in C_r$, z' being situated inside C_s. Eq.(85) can be verified by the substitution into eq.(84). It can be proved that (85) possesses the required properties (63). It follows from (85) that the 3/2-forms $\Psi_{\sigma,N_s}(t')$ provided that z' is situated inside C_s can be written down as

$$\Psi_{\sigma,N_s} = \Psi^{(1)}_{\sigma,N_s} + \sum_{n \neq s} \hat{\Psi}^{(1)}_{\sigma,N_s n}(I - \hat{S}^{(1)}_\sigma)^{-1}_{ns} S^{(1)}_{\sigma,ss} \tag{86}$$

where $\Psi^{(1)}_{\sigma,N_s n}$ is the integral operator that performs the integration over t_1 along C_n-contour, its kernel being $\Psi^{(1)}_{\sigma,N_s}(t_1)dt_1$. In the next Section we use (85) and (86) to determine the partition functions in (15).

6 Calculation of the partition functions

Having the vacuum correlators to be known, in this Section we calculate from eq.(41) the partition functions $\hat{Z}^{(n)}_{L,L'}$ in eq.(15). We continue this calculation in the next Section VII where the final formulae for the partition functions will be given.

To solve eq.(41) it is useful to remove the dependence on t in (41) employing for this aim the following relations

$$\sum_{p=a,b} \int_{C_p^{(r)}} \tilde{\chi}_{N_s}(t)\frac{d\theta dz}{2\pi i} Y_{b,N_s}(t) = \delta_{N_s N_r} \quad \text{for} \quad (N_r, N_s) \in \{N\}. \tag{87}$$

In (87) the $\{N\}$ set is associated with those Schottky parameters, which are moduli. To obtain (87) we represent $\tilde{\chi}_{N_s}(t)$ in the form of the integrals along the infinitesimal contour $C(z)$ surrounding z-point. The above contour is transformed to obtain the sum of the integrals along the C_r contours that reduced to (87). For this aim we use eq.(C14) of Appendix C. Below we also will use the kindred relations for both $\chi_{N_s}(t)$ in (63) and $\Psi_{\sigma,N_s}(t)$ in (75). The above relations are given by

$$\sum_{p=a,b} \int_{C_p^{(r)}} \chi_{N_s}(t)\frac{d\theta dz}{2\pi i} Y_{b,N_s}(t) = \delta_{N_s N_r} \quad \text{and} \quad \int_{C_b^{(r)}} \frac{d\theta dz}{2\pi i} \Psi_{\sigma,N_s}(t)\hat{Y}^{(1)}_{\sigma,N_s}(t) = \delta_{N_s,N_r}, \tag{88}$$

see Appendix C for more details. Unlike (87), (88) are true for all N_s including the $\{N_o\}$ set, as well.

By means of (87), we transform (41) to the set of equations

$$\partial q_{N_r} \ln \hat{Z}^{(n)}_{L,L'} = \sum_{p=a,b} \int_{C_p^{(r)}} \left[< T_{gh}(t) > + < T_m(t) > - \sum_N \frac{\partial}{\partial q_N} \tilde{\chi}_N(t) \right] dt Y_{p,N_r}(t) \tag{89}$$

where $dt = d\theta dz/2\pi i$. The summation over N in (89) is performed as in (41). As in (41), the derivatives with respect to odd moduli in eq.(89) are implied to be the "right" ones.

Both $< T_{gh}(t) >$ and $< T_m(t) >$ in (89) are calculated from (42) and (43) in the terms of the $G_{gh}(t,t')$ and $K(t,t')$ Green functions given in the previous Sections.

Eqs.(89) determine the partition functions up to an arbitrary factor independent of the moduli. But this factor may depend on n, L and L', as well. In addition, the discussed factor may depend on the $\{q_{N_o}\}$ set of those (3|2) Schottky parameters, which are chosen to be the same for all the genus-n supermanifolds and, therefore, they are not moduli. Below we choose the above $\{q_{N_o}\}$ set as $\{q_{N_o}\} = (u_1, v_1, u_2|\mu_1, \nu_1)$.

The dependence on the above (3|2) Schottky parameters determined from the condition that the superstring amplitudes are independent of a choice of these (3|2) parameters [11,12]. Moreover, the dependence on n, L and L' in the discussed factor is calculated from the condition of the supermodular invariance together with the factorization requirement when the handles move away from each other. So, only a coupling constant remains to be arbitrary. We write down the $\hat{Z}^{(n)}_{L,L'}$ partition functions as follows

$$\hat{Z}^{(n)}_{L,L'} = g^{2n}\left[\det 2\pi i[\overline{\omega(\{q_{N_s}\},L')} - \omega(\{q_{N_s}\},L)]\right]^{-5} Z^{(n)}_L(\{q_{N_s}\})\overline{Z^{(n)}_{L'}(\{q_{N_s}\})} \tag{90}$$

where g is a coupling constant, $\omega(\{q_{N_s}\},L)$ is the period matrix and $Z^{(n)}_L(\{q_{N_s}\})$ is the holomorphic function of the $\{q_{N_s}\}$ Schottky parameters. This holomorphic structure of $\hat{Z}^{(n)}_{L,L'}$ arises because the superstring modes can be divided into left and right movers.

The only non-holomorphic factor on the right side of (90) originates from the proportional to $[\overline{\omega(\{q_{N_s}\},L')} - \omega(\{q_{N_s}\},L)]^{-1}$ terms in the scalar superfield vacuum correlator (31). The contribution of these terms into the right side of (89) is

$$5[2\pi i(\overline{\omega(\{q_{N_s}\},L')} - \omega(\{q_{N_s}\},L))^{-1}]_{mn} \sum_{p=a,b} \int_{C_p^{(r)}} 2\pi i D(t)\eta_m(t;L) 2\pi i \eta_n(t;L) dt Y_{p,N_r}(t) \tag{91}$$

where $\eta_n(t;L)$ is the half-form (37). To obtain the discussed non-holomorphic factor in (90) we prove firstly that the integral in (91) is equal to $2\pi i \partial_{N_r}\omega_{mn}(\{q_{N_s}\},L)$. This proof employs the transformation low under the $\Gamma_{p,s}$ mappings ($p = a, b$) of $\partial_{N_s}F_q(t)$, which can be done provided that the above transformation low for $F_q(t)$ is known. To obtain the desired transformation low one is due only to express the $\partial_{N_s}F_q(\Gamma_{p,s}(t))$ derivatives taken for $\Gamma_{p,s}(t)$ to be constant in the terms of those calculated for t being constant. If $F_p \to Q^q_{\Gamma_{p,s}}\tilde{F}_q$ under the considered mapping, then we obtain for the derivative that

$$\partial_{N_s}F_q(t) \to Q^q_{\Gamma_{p,s}}(t)\left[\partial_{N_s}\tilde{F}_q(t) - E^{(q)}_{p,N_s}\left(\tilde{F}_q(t)\right)\right] \tag{92}$$

where $E^{(q)}_{p,N_s}(F(t)$ is defined by

$$E^{(q)}_{p,N_s}(F(t)) = \frac{q}{2}F(t)\partial_z Y_{p,N_s}(t) + \frac{\epsilon(F)}{2}[DF(t)]DY_{p,N_s}(t) + [\partial_z F(t)]Y_{p,N_s}(t). \tag{93}$$

In this equation D is the spinor derivative (27). Furthermore, $\epsilon(F) = 1$, if F_q obeys the fermion statistics and $\epsilon(F) = -1$ otherwise. To obtain the above formulae we use for the derivatives of $Y_{p,N_s}(t)$ eqs.(A6) of Appendix A.

Employing (92), in Appendix D we derive the representation of $\partial_{N_r} J_m(t)$ in the form of the integral along $C_a^{(r)}$ and $C_b^{(r)}$. The desired statement about the integral in (91) follows from this representation for $\partial_{N_r} J_m(t)$, as it is shown in Appendix D.

As far as the integral in (91) is found to be $2\pi i \partial_{N_r} \omega_{mn}(\{q_{N_s}\}, L)$, one concludes that (91) present none other than the derivative with respect to N_r of the non-holomorphic factor in (90). The above non-holomorphic factor already discussed in [23,30]. This non-holomorphic factor extends to depending on L period matrices the well known non-holomorphic multiplier [7] arising in the boson string theory.

To determine the holomorphic $Z_L^{(n)}(\{q_{N_s}\})$ factor in (89), it is convenient to integrate by parts the $\partial(FB) - D[(DF)B]/2$ terms in $< T_{gh} >$. Simultaneously we carry out the ∂q_N with $\tilde{\chi}_{N_s}$ to $Y_{p,N_s}(t)$. For this aim we take into account eqs.(87). The above procedure, as well as the integration by parts originates the out integral terms. These terms disappear owing to (A5) with the exception of those $I_{(an)}$ terms, which are due to the conformal anomaly in (FB) and in $(DF)B$. These $I_{(an)}$ terms will be canceled by kindred terms appearing in the following calculations. The resulting equations for $Z_L^{(n)}(\{q_{N_s}\})$ turn out to be

$$\partial_{q_{N_r}} \ln Z_L^{(n)}(\{q_{N_s}\}) = \sum_{p=a,b} \int_{C_p^{(r)}} dt W_{p,N_r}(t) + \sum_{p=a,b} \int_{C_p^{(r)}} dt W_{p,N_r}^{(o)}(t) + I_{(an)} \tag{94}$$

where $I_{(an)}$ are the out integral terms discussed above. Furthermore, $W_{p,N_r}(t)$ in (94) is defined as

$$W_{p,N_r}(t) = -5\partial_z K(t,t) Y_{p,N_r} + E_{p,N_r}^{(-2)}(G(t,t)) - \sum_N (-1)^{e(N)e(N_r)} \chi_N(t) \partial q_N Y_{p,N_r}(t) \tag{95}$$

where $E_{p,N_r}^{(-2)}$ is defined by (93) at $q = -2$ and the summation over N performs as in (89). Moreover, $e(N_r) = 1$ for $N_r = (\mu_r, \nu_r)$ and $e(N_r) = 0$ for $N_r = (k_r, u_r, v_r)$. In the accordance with the general prescription [25] both the first term and the second one in (95) are defined as the limit at $t' \to t$ of $-5\partial_z K(t,t') Y_{p,N_r} + E_{p,N_r}^{(-2)}(G(t,t'))$, the singular term $(\theta - \theta')(z - z')^{-1}$ being omitted in both $K(t,t')$ and $G(t,t')$. The Green function $K(t,t')$ appears in (95) instead of the vacuum correlator (31) of the scalar superfields because the difference of above (31) and $K(t,t')$ has been already taken into account by the above discussed non-holomorphic factor in (90). Besides, in (95) we use $G(t,t')$ instead of the ghost correlator $G_{gh}(t,t')$. The difference of $G_{gh}(t,t')$ and $G(t,t')$ being calculated from (62), contributes into (94) by the second term on the right side of (94) with $W_{p,N_r}^{(o)}(t)$ to be

$$W_{p,N_r}^{(o)}(t) = (-1)^{e(N_r)} \sum_{N_o,N_o'} \left[E_{p,N_o}^{(-2)}(Y_{b,N_r}(t)) + \sum_N [\partial q_N Y_{p,N_r}(t)] B_{NN_o} \right] A_{N_o N_o'}^{-1} \chi_{N_o'}(t) \tag{96}$$

where the $B_{N_s N_o}$ matrix is determined in (64) amd the $A_{N_o N_o'}$ matrix in (94) is the same as in (62).

In the following consideration we express $G(t,t')$ in (95) in the terms of $S_\sigma(t,t')$. For this aim we employ eq.(76). The integrals over $t_1=(z_1|\theta_1)$ in (76) give raise the integrals in (94) over both t and t_1. In these integrals the integration over t_1 is performed before the integration over t, but it will be convenient for the solution of eqs.(94) to change the above order of the integration and to perform the integration over t before the integration over t_1. In this case one is due to be careful with the terms associated with $s=r$ in the sum over s in (76). Indeed, the above terms contribute into (89) as

$$-\sum_{p=a,b}\sum_{N'_r}\int\limits_{C_p^{(r)}}(-1)^{e(N_r)e(N'_r)+e(N_r)}dt\chi_{N'_r}(t)E^{(-2)}_{p,N_r}\left(S_\sigma(t,t_1)\right)dt_1Y_{p,N'_r}(t_1) \tag{97}$$

where the integrations over both t and t_1 are performed along the same path. So, changing the order of the integration over t and t_1, one meets with ambiguities due to the pole at $z=z_1$ in $S_\sigma(t,t_1)$. To avoid these ambiguities, we note that in (97) the terms with $N_r=k_r$ are absent because of eqs.(C15) and also because $Y_{b,k_s}(t)=\tilde{Y}^{(1)}_{\sigma,k_s}(t)$ and $Y_{a,k_s}(t)=0$, as well. Therefore, in (97) the integrals over t_1 can be given in the form of the integrals over the C_r contour as follows

$$-\sum_{N''_r}\int\limits_{C_r}S_\sigma(t,t_1)dt_1Y_{p,N''_r}(t_1)A^{-1}_{N''_rN'_r} \tag{98}$$

where the C_r contour is defined in Sec.IV and Sec.V and $A_{N''_rN'_r}$ is the submatrix $B_{N''_rN'_r}$ of the $B_{N_sN_r}$ matrix (64). In (98) there is implied that the pole at$z_1=z$ is situated outside of the C_r contour on the z_1 complex plane. But we accommodate the above pole inside the C_r contour. At the same time, we add in (98) the suitable terms to cancel the appearing contribution of the discussed pole. In this case the desired change of the order of the integration in (97) can be performed without doubt, the result being

$$\sum_{N'_r,N''_r}\sum_{p=a,b}\int\limits_{C_r}dt_1Y_{p,N''_r}(t_1)A^{-1}_{N''_rN'_r}\int\limits_{C_p^{(r)}}\chi_{N'_r}(t)dtE^{(-2)}_{p,N_r}(S_\sigma(t,t_1))-$$

$$\sum_{N'_r,N''_r}\sum_{p=a,b}\int\limits_{C_p^{(r)}}E^{(-2)}_{p,N_r}(Y_{p,N''_r}(t))A^{-1}_{N''_rN'_r}\chi_{N'_r}(t)dt. \tag{99}$$

After the $(t_1\to t, t\to t_1)$ redefinition in (99), eqs.(94) turn out to be

$$\partial_{q_{N_r}}\ln Z^{(n)}_L(\{q_{N_s}\})=\sum_{p=a,b}\int\limits_{C_p^{(r)}}dtW^{(o)}_{p,N_r}(t)+\sum_{p=a,b}\int\limits_{C_p^{(r)}}dt\tilde{W}_{p,N_r}(t)+\int\limits_{C_r}dt\hat{W}_{N_r}(t)+I_{(an)} \tag{100}$$

where $W^{(o)}_{p,N_r}(t)$ is defined by (96) and $\tilde{W}_{p,N_r}(t)$ is given by

$$\tilde{W}_{p,N_r}(t)=-5\partial_zK(t,t)Y_{p,N_r}+E^{(-2)}_{p,N_r}\left(S_\sigma(t,t)\right)-\sum_N(-1)^{e(N)e(N_r)}\chi_N(t)\partial q_NY_{p,N_r}(t)-$$

$$\sum_{N'_r,N''_r}(-1)^{e(N_r)e(N'_r)+e(N'_r)}\chi_{N'_r}(t)E^{(-2)}_{p,N_r}(Y_{p,N''_r}(t))A^{-1}_{N''_rN'_r}. \tag{101}$$

In (101) both the first term and the second one are again defined as the limit at $t' \to t$ of $-5\partial_z K(t,t')Y_{p,N_r} + E^{(-2)}_{p,N_r}(S_\sigma(t,t'))$, the singular term $(\theta-\theta')(z-z')^{-1}$ being omitted in both $K(t,t')$ and $S_\sigma(t,t')$. Furthermore, $\hat{W}_{N_r}(t)$ in (100) is given by

$$\hat{W}_{N_r}(t) = \sum_{N'_r,N''_r} Y_{p,N'_r}(t_1)A^{-1}_{N''_r N'_r}\sum_{p=a,b}\int_{C_p^{(r)}}\chi_{N'_r}(t_1)dt_1 E^{(-2)}_{p,N_r}(S_\sigma(t_1,t)). \tag{102}$$

Eqs. (101) and (102) follow directly from (96) and (99).

One can verifies that in the expression contained in the big square brackets in (96) all those terms disappear, which are more higher degree in (z,θ), than degree-2. So the discussed expression is the degree-2 polynomial in (z,θ). Therefore, the calculation of the first term in (100) can be performed by using of the integral relations (88) for $\chi_{N_r}(t)$. In particulary, one obtains that in the $N_r = k_r$ case the discussed term in (100) is equal to zero. Indeed, one can check that in the $N_r = k_r$ case the functions in the big square brackets in (96) have no periods under both $\Gamma_{a,r}$ and $\Gamma_{b,r}$ mappings. Therefore, being the degree-2 polynomial, the above functions are proportional to $Y_{b,k_r}(t)$. It is because only $Y_{b,k_r}(t)$ among of the degree-2 polynomials has no periods under twists about (A_r, B_r)-cycles. So the discussed term in (100) is expressed in the terms of the integrals given in (88). And, as far as k_r are not contained in the $\{q_{N_o}\}$ set, one concludes that the considered term in (100) is equal to zero.

For $N_r \neq k_r$, one can verify that the discussed first term in (100) can be given as follows

$$\sum_{N''_r}\int_{C_r} dt W^{(o)}_{p,N''_r}(t)A^{-1}_{N''_r N_r} \tag{103}$$

where the C_r contour defined in Sec.IV and Sec.V, the $A_{N''_r N_r}$ matrix being the same as in (98) and $W^{(o)}_{p,N''_r}(t)$ is given by (96). Using (A3), we can present the degree-2 polynomials in the square brackets in (96) as the sum of the $Y_{bN_r}(t)$ polynomials. We use (64) to reduce the discussed integrals along the C_r contour to the integrals given in (88). The above reduction can be performed because the B matrix in (64)] is independent on p. So, the discussed first term in (100) are calculated without employing the explicit form of $\chi_{N_o}(t)$. The result is that the discussed first term on the right side of (100) originates in $Z^{(n)}_L(\{q_{N_s}\})$ only the $H_o(\{q_{N_o}\},\mu_2)$ factor as follows

$$H_o(\{q_{N_o}\},\mu_2) = 1 - \frac{\mu_1\mu_2}{2(u_1-u_2)} - \frac{\nu_1\mu_2}{2(v_1-u_2)}. \tag{104}$$

The above $H_o(\{q_{N_o}\},\mu_2)$ factor depends on the choice of the $\{q_{N_o}\}$ set of parameters that are not moduli. Eq.(104) is obtained provided that the $\{q_{N_o}\}$ set is chosen to be $(u_1, v_1, u_2|\mu_1,\nu_1)$.

To calculate the other terms on the right side of (100) we use for $K(t,t')$ and for $S_\sigma(t,t')$ eqs. (59) and (85), respectively. In the following, it will be convenient to divide $\tilde{W}_{p,N_r}(t)$ into the sum of three terms as

$$\tilde{W}_{p,N_r}(t) = W^{(r)}_{p,N_r}(t) - 5\partial_z[K(t,t) - K^{(1)}_s(t,t)]Y_{p,N_r} + W'_{p,N_r}(t) \tag{105}$$

where $K^{(1)}_s(t,t)$ is defined in the terms of $R^{(1)}_s(t,t)$ by (36). In turn, $R^{(1)}_s(t,t)$ is defined by (46). In the $\partial_z[K(t,t) - K^{(1)}_s(t,t)]$ term, the derivative are taken with respect to only the first

argument of both $K(t,t)$ and $K_s^{(1)}(t,t)$. Furthermore, $W^{(r)}_{p,N_r}(t)$ is given by

$$W^{(r)}_{p,N_r}(t)=\sum_{N'_r}\Psi^{(1)}_{\sigma,N'_r}(t)\left[E^{(-2)}_{p,N_r}\left(Y^{(1)}_{\sigma,N'_r}(t)\right)-\partial_{q_{N_r}}Y_{p,N'_r}(t)\right]-$$
$$5\partial_z K_r^{(1)}(t,t)Y_{p,N_r}+E^{(-2)}_{p,N_r}(S^{(1)}_{\sigma,r}(t,t)) \tag{106}$$

where $Y^{(1)}_{\sigma,N'_r}(t)$ is the same as in (75). Moreover, $S^{(1)}_{\sigma,r}(t,t)$ and$\Psi^{(1)}_{\sigma,N'_r}(t)$ are defined by (81) and (82). In (106) the $-5\partial_z K_r^{(1)}(t,t)Y_{p,N_r}+E^{(-2)}_{p,N_r}(S^{(1)}_{\sigma,r}(t,t))$ term is again defined as the limit at $t'\to t$ of $-5\partial_z K_r^{(1)}(t,t')Y_{p,N_r}+E^{(-2)}_{p,N_r}(S^{(1)}_{\sigma,r}(t,t'))$, the singular term $(\theta-\theta')(z-z')^{-1}$ being omitted in both $K_r^{(1)}(t,t')$ and $S^{(1)}_{\sigma,r}(t,t')$.

Furthermore, $W'_{p,N_r}(t)$ in (105) is the rest of $\tilde W_{p,N_r}(t)$ that remains after the subtraction of two first terms on the right side of(105). So

$$W'_{p,N_r}(t)=\tilde W_{p,N_r}(t)-W^{(r)}_{p,N_r}(t)+5\partial_z[K(t,t)-K_r^{(1)}(t,t)]Y_{p,N_r} \tag{107}$$

where $\tilde W_{p,N_r}(t)$ is given by (101) and the $5\partial_z[K(t,t)-K_r^{(1)}(t,t)]Y_{p,N_r}$ term is defined as in (105).

The proportional to $\Psi^{(1)}_{\sigma,N'_r}(t)$ terms are added into (106) and, at the same time, they are subtracted from $W'_{p,N_r}(t)$. So $\tilde W_{p,N_r}(t)$ remains unchanged. In is case the sum of the $W^{(r)}_{p,N_r}(t)$ contribution to (100) and of the I_{an} term has the form of the derivative with respect to N_r of a function of the moduli. So the considered terms in (100) originate in $Z_L^{(n)}(\{q_{N_s}\})$ some multiplier $\tilde Z^{(1)}(q_{N_r};l_{1r},l_{2r})$. Simultaneously, $W'_{p,N_r}(t)$ together with $\hat W_{p,N_r}(t)$ originates in $Z_L^{(n)}(\{q_{N_s}\})$ the factor $\tilde Z^{(n)}_{gh}(\{q_{N_s}\},L)$ that satisfies the following equation

$$\partial_{q_{N_r}}\ln\tilde Z^{(n)}_{gh}(\{q_{N_s}\},L)=\sum_{p=a,b}\int\limits_{C_p^{(r)}}dtW'_{p,N_r}(t)+\int\limits_{C_r}dt\hat W_{p,N_r}(t) \tag{108}$$

where $\hat W_{p,N_r}(t)$ is defined by (102).

In the calculation of the $\tilde Z^{(1)}(q_{N_r};l_{1r},l_{2r})$ factor that is due to (106), we consider separately those of the proportional to $\Psi^{(1)}_{\sigma,N'_r}(t)$ terms, which do not contain the derivatives of the $J^{(1)}_{(o)r}(z_r)$ functions. The above derivatives appear, in general, because both $Y^{(1)}_{\sigma,\mu_r}(t)$ and $Y^{(1)}_{\sigma,\nu_r}(t)$ may be proportional to $\exp[\pm J^{(1)}_{(o)r}]$. In the discussed terms the polynomial in (z,θ) factors appear to be not higher degree, than 2, just as in (96). We again can use (A3) to represent the above polynomial factors as a sum over $\hat Y^{(1)}_{\sigma,N_s}(t)$. So the contribution of the considered terms into (100) can be calculated by using of relations (88) without the knowledge of the explicit form of $\Psi^{(1)}_{\sigma,N'_r}(t)$.

To calculate the contribution into (100) of the remainder of the proportional to $\Psi^{(1)}_{\sigma,N'_r}(t)$ terms, one needs the explicit form of the sum over N'_r of $Y^{(1)}_{\sigma,N'_r}(t)\Psi^{(1)}_{\sigma,N'_r}(t)$. This value can be found from (82). It is convenient to calculate the considered terms together with two last terms on the right side of (106). Moreover, it is useful to employ the t_s supercoordinates (23) instead of t. Under this substitution the additional terms arise that are due to the conformal

anomaly in (FB) and in $(DF)B$. These terms cancel the $I_{(an)}$ term in (100). The desired $\tilde{Z}^{(1)}(q_{N_r};l_{1r},l_{2r})$ factor turns out to be

$$\tilde{Z}^{(1)}(q_{N_r};l_{1r},l_{2r})=\frac{k_r^{\sigma_r l_{1r}}H_{(1)}(q_{N_r})[1+(-1)^{l_{2r}}k_r]^{2l_{1r}}}{k_r^{(3-2l_{1r})/2}[1+(-1)^{l_{2r}}k_r^{1/2}]^{4l_{1r}}}Z^{(1)}(k_r;l_{1r},l_{2r}) \tag{109}$$

where the super-Schottky multipliers k_s are defined in (19) and $Z^{(1)}(k;l_1,l_2)$ is

$$Z^{(1)}(k;l_1,l_2)=\prod_{p=1}^{\infty}\frac{[1+(-1)^{2l_2}k^pk^{(2l_1-1)/2}]^8}{[1-k^p]^8}. \tag{110}$$

In (109) and (110) it is implied that $l_j\in(0,1/2)$ where $j=1,2$. The $H_{(1)}(q_{N_r})$ factor in (109) is

$$H_{(1)}(q_{N_r})=(u_r-v_r-\mu_r\nu_r)^{-1}. \tag{111}$$

The terms in the square brackets in (105) originate in the considered $Z_L^{(n)}(\{q_{N_s}\})$ the multiplier $Z_m^{(n)}(\{q_{N_s}\},L)$. The above multiplier turns out to be

$$\ln Z_m^{(n)}(\{q_{N_s}\},L)=-5trace\ln(I-\hat{K}^{(1)}+\hat{\varphi}\hat{f}) \tag{112}$$

where the $\hat{K}^{(1)}$ and $\hat{\varphi}\hat{f}$ operators are the same as in (58). For superspin structures without the odd genus-1 superspin ones eq.(112) is proved in a rather simple manner. In this case the discussed factor is determined by the second term in (61). As far as $\hat{K}_{ss}^{(1)}=0$, the above term in (61) for $r=s$ being considered, can be rewritten as $\hat{K}^{(1)}\hat{K}\tilde{K}^{(1)}$. So the above term contributes to eqs.(100) as follows

$$-5\sum_{m_1\neq r}\int_{C_{m_1}}dt_1\sum_{m_2}\int_{C_{m_2}}\hat{K}(t_1,t_2)dt_2\sum_{p=1,2}\int_{C_p^{(r)}}K_r^{(1)}(t_2,t)dt\partial_zK_r^{(1)}(t,t_1)Y_{N_r}(t) \tag{113}$$

where the C_m contours are defined as in (53). The integral over t in (113) can be compared with the following expression for $\partial_{N_r}K_r^{(1)}(t_2,t_1)$:

$$\partial_{N_r}K_r^{(1)}(t_2,t_1)=\sum_{p=1,2}\int_{C_p^{(r)}}K_r^{(1)}(t_2,t)dtE_{p,N_s}^{(0)}(K_r^{(1)}(t,t_1)) \tag{114}$$

where $E_{p,N_s}^{(0)}$ is given by (93) at $q=0$. To obtain eq.(114) we represent $\partial_{N_r}K_r^{(1)}(t_2,t_1)$ in the form of the integral over t along the infinitesimal contour $C(z_2)$ that surrounds the z_2 point, the integrand being $K_r^{(1)}(t_2,t)\partial_{N_r}K_r^{(1)}(t,t_1)$. Using (92), we transform the above integral to the integral (114) along $C_a^{(r)}$ and $C_b^{(r)}$ contours, see also Appendix D.

We argue that the difference of (114) and the integral over t in (113) does not contributes in (113). The simplest way to prove this statement is to add in $<T_m>$ the term $5D[<(DX)(DX)>]$. The above term is equal to zero because DX obeys the fermi statistics. Being substituted in (89) and integrated by parts, the discussed term transforms the integral over t in

(113) into the integral (114). So the integral over t in (113) can be replaced by $\partial_{N_r} K_r^{(1)}(t_2, t_1)$. So (113) appears to be $-5\partial_{N_r} trace \ln(I - \hat{K}^{(1)})$ that proves eq.(112).

The proof of eq.(112) for those even genus-n superspin structures where the odd genus-1 superspin structures present can be done in a like manner. In this case the discussed $Z_m^{(n)}(\{q_{N_s}\}, L)$ factor is determined by all the terms in (61) except only the first term.To obtain the integral representations of the derivatives with respect to the moduli we use (D20) and (D21) of Appendix D. The contribution into eqs.(100) due to the discussed terms turns out to be

$$5trace \left\{ \left[I + \hat{K} - (I + \hat{K})\varphi V^{-1} \hat{f} - (I + \hat{K})\varphi V^{-1} \hat{f} \right] \partial_{N_r}(-\hat{K}^{(1)} + \hat{\varphi} \hat{f}) \right\}. \tag{115}$$

On the other side, one can verify that

$$(I - \hat{K}^{(1)} + \hat{\varphi}\hat{f})^{-1} = I + \hat{K} - (I + \hat{K})\hat{\varphi} V^{-1} \hat{f} - (I + \hat{K})\hat{\varphi} V^{-1} \hat{f}. \tag{116}$$

So eq.(115) can be rewritten as $-5\partial_{N_r} trace \ln(I - \hat{K}^{(1)} + \hat{\varphi}\hat{f})$ that proves eq.(112).

The calculation of $\tilde{Z}_{gh}^{(n)}(\{q_{N_s}\}, L)$ from eqs.(108) is performed by the kindred method. We use (D22) and (D23) of Appendix D. We use also (88) and (87) for the calculation of the appropriate integrals. Moreover, in this calculation a number of terms disappears owing to the following identity

$$\sum_{N'_r} \sum_{p=1,2} \int\limits_{C_p^{(r)}} \chi_{N'_r}(t) dt \left[E_{p,N_r}^{(-2)}(Y_{p,N'_r}(t)) - \partial_{q_{N_r}} Y_{p,N'_r}(t) + (-1)^{e(N'_r)e(N_r)} \partial_{q_{N'_r}} Y_{p,N_r}(t) \right] = 0 \tag{117}$$

To prove (117) one uses that in (117), just as in (96), the polynomials in the square brackets are not higher degree in (z, θ), than degree-2. So, to calculate the left side of (117), we again can use (A3) and (88). The most simple way to perform this calculation is to go from t to the t_r variables (23). As the result, eq.(117) arises. The result of the calculation of $\tilde{Z}_{gh}^{(n)}(\{q_{N_s}\}, L)$ is given as follows

$$\ln \tilde{Z}_{gh}^{(n)}(\{q_{N_s}\}, L) = \ln sdet[M(\{\sigma_p\})] + trace \ln(I - \hat{S}_\sigma^{(1)}) \tag{118}$$

where the $M(\{\sigma_p\})$ matrix is defined in (77), the $\hat{S}_\sigma^{(1)}$ operator is the same as in (84) and the superdeterminant $sdet\tilde{U}$ of any $\tilde{U}$ matrix is defined as

$$sdet\tilde{U} = \frac{\det \tilde{U}_{(bb)}}{\det \tilde{U}_{(ff)}} \det[I - \tilde{U}_{(bb)}^{-1} \tilde{U}_{(bf)} U_{(ff)}^{-1} \tilde{U}_{(fb)}] \tag{119}$$

where $\tilde{U}_{(bb)}$, $\tilde{U}_{(bf)}$, $\tilde{U}_{(fb)}$ and $\tilde{U}_{(ff)}$ are submatrices forming the above $\tilde{U}$ matrix. The index b labels boson components and the index f labels the fermion ones. The $M_{N_s N'_r}(\{\sigma_p\})$ element of the $M(\sigma_p)$ matrix (77) are found to have the following form

$$M_{N_s N'_r}(\{\sigma_p\}) = \delta_{rs} \tilde{I}_{N_s} M_{(1)N_s N'_r}(\sigma_s) + M'_{N_s N_r}(\{\sigma_p\}) \tag{120}$$

where $M'_{N_s N_r}(\{\sigma_p\}) \to 0$, if the handle labeled by s (or by r) goes to infinity provided that the above handle is associated with the even genus-1 spin structure. Furthermore, the $\tilde{I}_{N_s}$ is defined in (120) as

$$\tilde{I}_{\mu_s} = 1 + 2l_{1s} - 2l_{1s}l_{2s}[1 - \sigma_s], \quad \tilde{I}_{\nu_s} = 1 + 2l_{1s} - 2l_{1s}l_{2s}[1 + \sigma_s]$$
$$\text{and} \quad \tilde{I}_{N_s} = 1 \quad \text{for} \quad N_s = k_s, u_s, v_s. \tag{121}$$

The $M_{(1)N_sN'_r}(\sigma_s)$ matrix in (120) has the following form

$$M_{(1)N_sN'_r}(\sigma_s)=\delta_{rs}M^{(s)}_{N_sN'_s}(\sigma_s) \tag{122}$$

The $M^{(s)}(\sigma_s)$ matrix in (122) is formed by the $M^{(s)}_{(ff)}(\sigma_s), M^{(s)}_{(bf)}(\sigma_s), M^{(s)}_{(fb)}(\sigma_s)$ and $M^{(s)}_{(ff)}(\sigma_s)$ submatrices. The index b labels boson components and the index f labels the fermion ones. The above submatrices are defined as follows. Firstly, $M^{(s)}_{(bb)}(\sigma_s)=I$ and $M^{(s)}_{(fb)}(\sigma_s)=0$. Besides, the elements of the $M^{(s)}_{(bf)}(\sigma_s)$ matrix are given by

$$M^{(s)}_{b\mu_s}(\sigma_s)=\frac{(2+\sigma)M^{(s)}_{b\mu_s}(\sigma_s)}{2-\sigma}=-\frac{l_{1s}(\mu_s-\nu_s)(1+2\sigma_s)}{2}\quad \text{for}\quad b=u_s,v_s$$
$$\text{and}\quad M^{(s)}_{k_sf}(\sigma_s)=0. \tag{123}$$

The elements $M_{ff'}(k_s,\sigma_s)$ of the above $M^{(s)}_{(ff)}(\sigma_s)$ matrix are defined as follows

$$\begin{aligned}&M_{\mu\mu}(k,-1)=M_{\mu\nu}(k,-1)=M_{\nu\nu}(k^{-1},1)=M_{\nu\mu}(k^{-1},1)=M_\mu(k)\quad\text{and}\\&M_{\nu\nu}(k,-1)=-3M_{\nu\mu}(k,-1)=M_{\mu\mu}(k^{-1},1)=-3M_{\mu\nu}(k^{-1},1)=M_\nu(k)\end{aligned} \tag{124}$$

where $M_\mu(k)$ and $M_\nu(k)$ are given by

$$M_\mu(k)=1-2l_{1s}+\frac{(-1)^{2l_{2s}}l_{1s}\sqrt{k}}{[1+(-1)^{2l_{2s}}\sqrt{k}]}\quad\text{and}\quad M_\nu(k)=1-2l_{1s}+\frac{3l_{1s}[1+(-1)^{2l_{2s}}k])}{[1+(-1)^{2l_{2s}}\sqrt{k}]}. \tag{125}$$

In (124) and (125) the index s is omitted. In (121)-(125) it is implied that $l_{1s}\in(0,1/2)$ and $l_{2s}\in(0,1/2)$. In the following consideration it will be convenient to rewrite (118) as

$$\tilde Z^{(n)}_{gh}(\{q_{N_s}\},L)=sdet[M(\{\sigma_p\})U^{-1}(\{\sigma_p\})]Z^{(n)}_{gh}(\{q_{N_s}\},L) \tag{126}$$

where $Z^{(n)}_{gh}(\{q_{N_s}\},L)$ is defined by

$$\ln Z^{(n)}_{gh}(\{q_{N_s}\},L)=trace\ln(I-\hat S^{(1)}_\sigma)+\ln sdet[U(\{\sigma_p\})]. \tag{127}$$

The $U(\{\sigma_p\})$ matrix in (126) and in (127) is defined as

$$U(\{\sigma_p\})=M(\{\sigma_p\})M^{-1}_{(1)}(\{\sigma_p\}). \tag{128}$$

In (128) the $M_{(1)}(\{\sigma_p\})$ matrix is defined by (122). One can verify that the $U_{N_s,N_r}(\{\sigma_p\})$ elements of the $U(\{\sigma_p\})$ matrix (128) have the following form

$$U_{N_s,N_r}(\{\sigma_p\})=\tilde I_{N_s}\delta_{N_sN_r}+\hat U_{N_s,N_r}(\{\sigma_p\}) \tag{129}$$

where $\hat U_{N_s,N_r}(\{\sigma_p\})$ decrease, if at least one of two handles labeled by s and by r goes away to infinity provided that the above handle is associated with the even genus-1 spin structure.. And $\tilde I_{N_s}$ is given by (121).

Collecting together all the obtained factors in $Z_L^{(n)}(\{q_{N_s}\})$, we write down the desired $Z_L^{(n)}(\{q_{N_s}\})$ factor in (90) as

$$Z_L^{(n)}(\{q_{N_s}\}) = \tilde{Z}^{(n)}(\{q_{N_s}\}, L) H(\{q_{N_s}\}) \prod_{s=1}^{n} \frac{(-1)^{2l_{1s}+2l_{2s}-1} 16^{2l_{1s}} Z^{(1)}(k_s; l_{1s}, l_{2s})}{k_s^{(3-2l_{1s})/2}} \tag{130}$$

where $Z^{(1)}(k_s; l_{1s}, l_{2s})$ is given by (110) and $\tilde{Z}^{(n)}(\{q_{N_s}\}, L)$ is defined by

$$\tilde{Z}^{(n)}(\{q_{N_s}\}, L) = Z_m^{(n)}(\{q_{N_s}\}, L) Z_{gh}^{(n)}(\{q_{N_s}\}, L) \tag{131}$$

where $Z_m^{(n)}(\{q_{N_s}\}, L)$ is defined by (112) and $Z_{gh}^{(n)}(\{q_{N_s}\}, L)$ is given by (127). Furthermore, the $H(\{q_{N_s}\})$ factor in (130) is defined as

$$H(\{q_{N_s}\}) = (u_1 - u_2)(v_1 - u_2) \left[1 - \frac{\mu_1\mu_2}{2(u_1 - u_2)} - \frac{\nu_1\mu_2}{2(v_1 - u_2)}\right] \prod_{s=1}^{n} (u_s - v_s - \mu_s\nu_s)^{-1} \tag{132}$$

In (132) it is assumed that u_1, v_1, u_2, μ_1 and ν_1 are fixed to be the same for all the genus-n supermanifolds and, therefore, they are not the moduli. Apart of the normalization factor $(u_1 - u_2)(v_1 - u_2)$, eq. (132) presents product H_o and H_1 factors defined by (104) and (111). As it has been noted already, the above factor is calculated [11,12] from the condition that the superstring amplitudes are independent of a choice of $\{q_{N_o}\}$ set. The u_1, v_1, u_2, μ_1 and ν_1 fixed parameters can be changed by fractionally linear supersymmetrical transformations (18). In (130) only the H factor (132) is changed under these transformations. The condition that the change of H is compensated by the changes of the differentials in (15) just gives the above normalization factor in (132) . In the calculation of the discussed normalization factor one is due to take into account that the above transformations depend not only on u_1, v_1, u_2, μ_1 and ν_1, but also on μ_2, which is the variable of the integration in (15).

In (130) the normalization factors $(-1)^{2l_{1s}+2l_{2s}-1}16^{2l_{1s}}$ are taken into account. To derive these factors in the case when $l_{1s}l_{2s} = 0$ we move away the handle labeled by the s index. In this case the genus-n amplitude is due to be proportional to the one loop amplitude [17]. One can verify that in the $l_{1s}l_{2s} = 0$ case the dependence on q_{N_s} Schottky parameters disappears in both (112) and (118). The comparison (130) with [17] gives the above normalization factors in (130).

Eq.(130) implies also the 16^2 factor for every pair of the odd genus-1 spin structures presenting in the given genus-n superspin structure. To derive this factor one can note that the superspin structure containing a pair of the odd genus-1 spin structures labeled, say, by the r and s indices, can be derived by the supermodular transformation of the superspin structure containing a pair of the handles with $l_{1r} = l_{12} = 1/2$ and $l_{2r} = l_{2s} = 0$. In the case of zero odd Schottky parameters the above transformation implies only adding $\pm 2\pi$ to the phase of $u_r - u_s, v_r - v_s, u_r - v_s$ or of $v_r - u_s$. In the case of arbitrary odd parameters that must be considered to derive the desired factors in (130), the discussed transformation includes, in addition, a change of the Schottky parameters. So the Jacobian in (15) arises under the discussed transformation. Fortunately, to derive the normalization factors in (130) one can consider the case when both k_1 and k_2 tend to zero. In this limit the discussed supermodular transformation

does not change of the Schottky parameters and, therefore in the considered limit the Jacobian of this transformation is equal to unity. In this case the requirement of the invariance of (15) under the considered supermodular transformation gives the above 16^2 factor announced in (130).

Though $Z_{gh}^{(n)}(\{q_{N_s}\},L)$ in (131) being given by (127), is formed by the multipliers depending on a choice of the $\{\sigma_p\}$ set, the above $Z_L^{(n)}(\{q_{N_s}\})$ factor does not depend on $\{\sigma_p\}$. Indeed, as it has been shown in Sec.V, the ghost vacuum correlator (62) is determined uniquely and, therefore, this correlator is independent of a choice of the $\{\sigma\}$ set. So, $Z_L^{(n)}(\{q_{N_s}\})$ is independent of a choice of $\{\sigma\}$. Therefore, one concludes that $Z_{gh}^{(n)}(\{q_{N_s}\},L)$ is also independent of $\{\sigma_p\}$.

Eq.(130) gives the desired $Z_L^{(n)}(\{q_{N_s}\})$ holomorphic multiplier in the partition function (90). To obtain the final result, in the next Section we yet calculate both $Z_{gh}^{(n)}(\{q_{N_s}\},L)$ and $Z_m^{(n)}(\{q_{N_s}\},L)$ multipliers in (131) in the terms of moduli and of Green functions.

7 Final expressions for the superstring amplitudes

To obtain the explicit formula for the $\tilde Z^{(n)}(\{q_{N_s}\},L)$ factor in (130) we rewrite this factor as

$$\tilde Z^{(n)}(\{q_{N_s}\},L) = \tilde Z_0^{(n)}(\{q_{N_s}\},L)\Upsilon_m^{(n)}(\{q_{N_s}\},L)\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L) \tag{133}$$

where $\tilde Z_0^{(n)}(\{q_{N_s}\},L)$ is the value of the considered factor at zero odd Schottky parameters. Moreover, $\ln\Upsilon_m^{(n)}(\{q_{N_s}\},L)$ and $\ln\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L)$ present the terms proportional to the odd Schottky parameters in (112) and in (127), respectively. So $\ln\Upsilon_m^{(n)}(\{q_{N_s}\},L)$ can be given as

$$\ln\Upsilon_m^{(n)}(\{q_{N_s}\},L) = \ln\left[1-\delta\hat K^{(1)}+\delta[\hat\varphi\hat f]+\Delta_m\right] \tag{134}$$

where $\delta\hat K^{(1)}$ and $\delta[\hat\varphi\hat f]$ denotes those terms in $\hat K^{(1)}$ and $\hat\varphi\hat f$, respectively, which are proportional to the odd Schottky parameters. Furthermore, the Δ_m integral operator is formed by the $\{\Delta_m^{(p)}\}$ set of the $\Delta_m^{(p)}$ integral operators, the "kernels" being $\Delta_m^{(p)}(t,t')dt'$. As in (59), we define the kernel together with the differential dt'. Every the $\Delta_m^{(p)}$ integral operator being applied to a function of t', performs integrating over t' along the C_p contour. The C_p contours are the same as in (56). Moreover, one can verify that

$$\begin{aligned}\Delta_m^{(p)}(t,t') = \int\limits_{C_p}\left[K_{(o)}(t,t_1)+\int\limits_{C_p}2K_{(o)}(t,t_2)dt_2\varphi_{(o)p}(t_2)f_{(o)p}(t_1)\right]dt_1\delta[\varphi_p(t_1)f_p(t')] - \\ \sum_{r\neq p}\int\limits_{C_r}\left[K_{(o)}(t,t_1)+\int\limits_{C_r}2K_{(o)}(t,t_2)dt_2\varphi_{(o)r}(t_2)f_{(o)r}(t_1))\right]dt_1\delta K_r^{(1)}(t_1,t')\end{aligned} \tag{135}$$

where $K_{(o)}(t,t')$, $\varphi_{(o)r}$ and $f_{(o)r}$ denote, respectively, $K(t,t')$, φ_r and f_r calculated at all odd Schottky parameters to be equal to zero. Eq.(135) follows from (59) and from (116).

Furthermore, $\ln\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L)$ in (133) can be given in a quite like manner in the terms of both the $S_{(o)\sigma}(t,t')$ Green function (78) and the proportional to odd parameter terms in $\hat{S}_\sigma^{(1)}$. Besides, in $\ln\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L)$ the $\ln sdet[U(\{\sigma_p\})U_{(o)}^{-1}(\{\sigma_p\})]$ term presents, as it follows from (127). The $U_{(o)}(\{\sigma_p\})$ is defined to be $U(\{\sigma_p\})$ at zero odd moduli. The fermion part $S_{(f)\sigma}(t,t')$ of $S_{(o)\sigma}(t,t')$ is calculated in the terms of the $G_{(\sigma)}(z,z')$ Green function (69) by (79). So $\ln\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L)$ turns out to be given in the terms of the following $G^0(t,t';\{\sigma_p\})$ Green function

$$G^0(t,t';\{\sigma_p\}) = G_b(z,z')\theta' + \theta G_{(\sigma)}(z,z') \tag{136}$$

where $G_b(z,z')$ is defined by (68) and $G_{(\sigma)}(z,z')$ is defined by (69). The final result for $\ln\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L)$ is found to be

$$\begin{aligned}\ln\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L) = trace\ln\left[I-\delta\hat{S}_\sigma^{(1)}+\Delta_{gh}(\{\sigma_p\})\right] + \\ \ln sdet[U(\{\sigma_p\})U_o^{-1}(\{\sigma_p\})U'(\{\sigma_p\})]\end{aligned} \tag{137}$$

where $\delta\hat{S}_\sigma^{(1)}$ is referred to those terms in $\hat{S}_\sigma^{(1)}$, which are proportional to the odd Schottky parameters. Moreover,$U_{(o)}(\{\sigma_p\})$ is $U(\{\sigma_p\})$ at zero odd moduli. Furthermore, the $\Delta_{gh}(\{\sigma_r\})\}$ integral operator is formed by the $\{\Delta_{gh}^{(p)}(\{\sigma_r\})\}$ set of the $\Delta_{gh}^{(p)}(\{\sigma_r\})$ integral operators, the kernels being $\Delta_{gh}^{(p)}(\{\sigma_r\})(t,t')dt'$. We again define the kernel together with the differential dt'. Every the $\Delta_{gh}^{(p)}(\{\sigma_r\})$ integral operator being applied to a function of t', performs integrating over t' along the C_p contour that is the same as in (56). Furthermore, the $U'_{N_rN_s}(\{\sigma_p\})$ elements of the $U'(\{\sigma_p\})$ matrix are defined as

$$\begin{aligned}U'_{N_rN_s}(\{\sigma_p\}) = I + \sum_{p\neq q}\int\limits_{C_p}\Psi_{\sigma,N_r}^{(0)}(t)dt\int\limits_{C_q}\delta S_{\sigma,p}^{(1)}(t,t_1)dt_1\times \\ \sum_h\int\limits_{C_h}\tilde{\Delta}^{(h)}(t_1,t_2)dt_2\int\limits_{C_s}G^0(t_2,t';\{\sigma_p\})dt'\hat{Y}_{N_s}^{(1)}(t')\end{aligned} \tag{138}$$

where $\Psi_{\sigma,N_r}^{(0)}(t)$ are 3/2-supertensors defined by (71). And $\tilde{\Delta}^{(h)}(t_1,t_2)dt_2$ present the kernels of the $\tilde{\Delta}^{(h)}$ integral operators. The $\{\tilde{\Delta}^{(h)}\}$ set of these operators forms the $\tilde{\Delta}$ operator that can be given as

$$\tilde{\Delta} = [I+\Delta_{gh}(\{\sigma_p\})]^{-1} \tag{139}$$

where the $\Delta_{gh}(\{\sigma_p\})$ operator is the same as in (137). The kernels $\Delta_{gh}^{(p)}(\{\sigma_r\})(t,t')dt'$ of the above $\Delta_{gh}(\{\sigma_p\})$ operator are defined by

$$\Delta_{gh}^{(p)}(\{\sigma_r\})(t,t') = -\sum_{r\neq p}\int\limits_{C_r}G^0(t,t_1;\{\sigma_q\})dt_1\delta S_\sigma^{(1)}(t_1,t'). \tag{140}$$

Eqs. (134)-(140) allow to calculate both $\ln\Upsilon_m^{(n)}(\{q_{N_s}\},L)$ and $\ln\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L)$, at least in the form of the series over the odd Schottky parameters.

So, to obtain the explicit formulae for the holomorphic factors in the partition functions (90) one is due to calculate at zero odd Schottky parameters the $\tilde{Z}^{(n)}(\{q_{N_s}\},L)$ factor in (130).

This factor is referred as $\tilde{Z}_0^{(n)}(\{q_{N_s}\},L)$ in (133). We show now that in the case of zero all the odd Schottky parameters the discussed holomorphic factors in (90) can be calculated explicitly. The above holomorphic factors at zero odd Schottky parameters being known, one can derive the desired $\tilde{Z}_0^{(n)}(\{q_{N_s}\},L)$ factor in (133). So the resulting holomorphic factors in (90) can be given as follows

$$Z_L^{(n)}(\{q_{N_s}\}) = Z_{0(m)}^{(n)}(\{k_s,u_s,v_s\},L)Z_{0(gh)}^{(n)}(\{k_s,u_s,v_s\},L)\times \\ H(\{q_{N_s}\})\Upsilon_m^{(n)}(\{q_{N_s}\},L)\Upsilon_{gh}^{(n)}(\{q_{N_s}\},L) \tag{141}$$

where $H(\{q_{N_s}\})$ is given by (132), $\Upsilon_m^{(n)}(\{q_{N_s}\},L)$ is defined by (134) and $\Upsilon_m^{(n)}(\{q_{N_s}\},L)$ is defined by (137). The $Z_{0(m)}^{(n)}(\{k_s,u_s,v_s\},L)$ factor in (141) is due to the string fields and $Z_{0(gh)}^{(n)}(\{k_s,u_s,v_s\},L)$ is due to the ghost fields. Both these factors being calculated at zero odd Schottky parameters, depend only on the even Schottky ones. To calculate the discussed factors, we use eqs.(94) for $N_r = (k_r,u_r,v_r)$. In the integrand in (94), there are the terms proportional to the derivatives with respect to odd modui of the Y_{N_r} polynomials. We calculate the above derivatives using eqs.(A1) of Appendix A. After the discussed derivatives to be calculated, we take all the odd Schottky parameters in (94) to be equal to zero. In this case the $I_{(an)}$ term disappears. Moreover, the integrals over the $C_a^{(r)}$ paths vanish because $Y_{a,u_r} = Y_{a,v_r} = Y_{a,k_r} = 0$ at $\mu_r = \nu_r = 0$. The integral of $W_{p,N_r}^{(o)}(t)$ disappears, too. It is useful to note that at zero odd Schottky parameters the boson field contributions and the fermion field contributions can calculate separately from each other. For the boson field contributions one can use the boson string calculations [10,13,26], but the fermion field contributions need in a special consideration.

In the calculation of $Z_{0(gh)}^{(n)}(\{k_s,u_s,v_s\},L)$ we express the $G(t,t')$ ghost Green function (63) in the terms of $\hat{G}^0(t,t';\{\sigma\})$ where the $\hat{G}^0(t,t';\{\sigma\})$ Green functions is defined as

$$\hat{G}^0(t,t';\{\sigma\}) = \frac{1}{2}[G^0(t,t';\{\sigma_p\}) + G^0(t,t';\{-\sigma_p\})] \tag{142}$$

where $G^0(t,t';\{\sigma_p\})$ are defined by (136). To employ $\hat{G}^0(t,t';\{\sigma\})$, it is much more convenient, than to employ $G^0(t,t';\{\sigma_p\})$ because in this case the terms proportional to σ_p disappear in the calculation of $Z_{0(gh)}^{(n)}(\{k_s,u_s,v_s\},L)$. To express $G(t,t')$ in the terms of $\hat{G}^0(t,t';\{\sigma\})$, we use (67), (73) and (136), as well. The integrals over z'' in (73) give raise the integrals in (94) over both t and t''. In these integrals the integration over t'' is performed before the integration over t, but we change the above order of the integration and perform the integration over t before the integration over t'', as we made it in Sec.VI above. In the terms associated with $s = r$ in the sum over s in (73) there is implied that the pole at$z'' = z$ is situated outside of the $C_b(r)$ contour on the z'' complex plane. But we accommodate the above pole inside the $C_b(r)$ contour adding simultaneously the suitable terms to cancel the contribution of the discussed pole.

In the right side of the discussed equations one observes a large number of terms, which can be rewritten in the form of the integral I' along the C_q contours as

$$I' = \frac{1}{2}\sum_{q=1}^{n}\int\limits_{C_q} dz[G_{(\sigma)}(z,z) - G_{(-\sigma)}(z,z)]\sum_s 2l_{1s}\sigma_s\partial_{q_{N_r}}J_{(o)s}(z) \tag{143}$$

where the C_q contours are defined in (56) and, as it is usual, $G_{(\sigma)}(z,z)$ is defined to be the limit of $G_{(\sigma)}(z,z')$ at $z\to z'$, the singular term $(z-z')^{-1}$ being omitted. The $G_{(\sigma)}(z,z')$ Green function is given by (73). Furthermore, $G_{(-\sigma)}(z,z)$ is obtained from $G_{(\sigma)}(z,z)$ by the $\sigma_p\to-\sigma_p$ replacement for every σ_p. The integrand in (143) has no singularities outside the C_q contours and, in addition, it tends to zero more rapidly than z^{-1} at $z\to\infty$. Therefore, $I'=0$. The rest of the terms on the right side of every of the considered equations can be written down in the form of the derivative with respect to q_{N_r} of a function of the moduli, which turns out to be the same for all the equations discussed. To verify this statement one needs the derivatives with respect to moduli of the multipliers assigned to group products of the basic Schottky group elements. The above derivatives can be obtained from the formulae given in Appendix E. The discussed expressions for the desired derivations have been already used in [10,13,26] though in [10,13,26] the above expressions were not given in an explicit form.

In the calculation of the $Z^{(n)}_{0(m)}(\{k_s,u_s,v_s\},L)$ factor in (141) we use the $R_{(o)}$ Green function (44). Furthermore, we employ the $Z^{(n)}_{0(m)}(\{k_s,u_s,v_s\},L_0)$ factor [11,12,22] assigned to the $L_0=\bigcup_s(l_{1s}=0,l_{2s}=1/2)$ superspin structure to express those contributions to every $Z^{(n)}_{0(m)}(\{k_s,u_s,v_s\},L)$, which do not depend on the superspin structure considered. In this case the desired $Z^{(n)}_{0(m)}(\{k_s,u_s,v_s\},L)$ factors can be written down as

$$Z^{(n)}_{0(m)}(\{k_s,u_s,v_s\},L)=\frac{\Theta^5[l_1,l_2](0|\omega^{(o)})}{\Theta^5[\{0\},\{1/2\}](0|\omega^{(o)})}\prod_{(k)}\prod_{m=1}^{\infty}\frac{(1-k^{m-1/2})^{10}}{(1-k^m)^{10}} \tag{144}$$

where Θ is the theta function. The Θ in the denominator associates with the S_0 spin structure. The period matrix $\omega^{(o)}$ being calculated at zero odd Schottky parameters, is given by eq.(B9) of Appendix B. The product over (k) is taken over all the multipliers of the Schottky group (17), which are not powers of other the ones. To obtain (144) we employ eq.(D18) of Appendix D for the derivatives with respect to the moduli of the period matrix. Besides, we use the equations of Appendix E for the derivatives with respect to moduli of the multipliers assigned to Schottky group products. In the calculation of $Z^{(n)}_{0(gh)}(\{k_s,u_s,v_s\},L)$ we employ also relations (87) and identity (117), as well. The desired $Z^{(n)}_{0(gh)}(\{k_s,u_s,v_s\},L)$ factor in (141) turns out to be

$$\begin{aligned}Z^{(n)}_{0(gh)}(\{k_s,u_s,v_s\},L)=&\frac{\exp[-\pi i\sum_{j,r}l_{1j}l_{1r}\omega^{(o)}_{jr}]}{\sqrt{\det M_{(o)}(\{\sigma_p\})\det M_{(o)}(\{-\sigma_p\})}}\left(\prod_{s=1}^{n}Z_0(k_s;l_{1s},l_{2s})\right)\times\\&\prod_{(k)}\prod_{m=1}^{\infty}\frac{(1-k^{m+1})^2}{[1-\Lambda(k,\{\sigma_p\})k^{m+1/2}][1-\Lambda(k,\{-\sigma_p\})k^{m+1/2}]}\end{aligned} \tag{145}$$

where, as in (144), the product over (k) is taken over all the multipliers of the Schottky group (17), which are not powers of the other ones and

$$\Lambda(k,\{\sigma_p\})=\exp\Omega_{\Gamma_{(k)}}(\{\sigma_p\}) \tag{146}$$

where $\Omega_{\Gamma_{(k)}}(\{\sigma_p\})$ is given by (70) for those group products of the basic Schottky transformations, which have the multiplier to be equal k. The $Z_0(k_s;l_{1s},l_{2s})$ factors in (145) are defined

by

$$Z_0(k_s; l_{1s}, l_{2s}) = \frac{(-1)^{2l_{1s}+2l_{2s}-1}(1-k_s)^2}{4^{2l_{1s}}k_s^{3/2}[1+(-1)^{2l_{2s}}\sqrt{k_s}k_s^{l_{1s}}]^{2-2l_{1s}}}. \tag{147}$$

Furthermore, the $M_{(o)}(\{-\sigma_p\})$ matrix in (145) is given by

$$M_{(o)}(\{\sigma_p\}) = \tilde{M}(\{\sigma_p\})M^{-1}_{(1,0)}(\{\sigma_p\}) \tag{148}$$

where the $\tilde{M}(\{\sigma_q\})$ matrix is defined by (74) and $M_{(1,0)}(\{\sigma_p\})$ is $M_{(1)}(\{\sigma_p\})$ taken at zero odd Schottky parameters. The above $M_{(1)}(\{\sigma_p\})$ matrix is the same as in (128). It is useful to remind that in both (144) and (144) the k multipliers are calculated at zero odd Schottky parameters. The $(-1)^{2l_{1s}+2l_{2s}-1}2^{-2l_{1s}}$ normalization factors in (147) is determined to obtain the $(-1)^{2l_{1s}+2l_{2s}-1}16^{2l_{1s}}$ normalization factors in (130).

Eqs.(141) being added by (134), (135) and (137)-(148), solve the problem of the calculation of the $Z_L^{(n)}(\{q_{N_s}\})$ holomorphic multipliers in the partition functions. The another form of the above holomorphic multipliers is presented by eqs. (130) together with (112), (127), (132) and (131). The discussed $Z_L^{(n)}(\{q_{N_s}\})$ holomorphic multipliers to be known, the partition functions are determined by (90). In this case the multi-loop amplitudes are calculated by (15). To calculate the vacuum expectations of the vertex products in (15) we use the $\hat{X}_{L,L'}(t,\overline{t};t',\overline{t}')$ vacuum correlator (31) of the scalar superfields. The above correlators are calculated in the terms of the holomorphic Green functions $R_L(t,t')$, the $J_r(t;L)$ functions and the period matrix $\omega(\{q_N\};L)$. At zero odd Schottky parameters, the above functions are given by (B8) and the period matrix is given by (B9). The depending on odd Schottky parameter terms in $J_r(t;L)$ and $\omega(\{q_N\};L)$ are taken into account by (54) and by (55). Furthermore, at zero odd Schottky parameters, the holomorphic Green function of the string fields is given by (44) together with (B7) and (45). The depending on odd Schottky parameter terms in the discussed holomorphic functions can be calculated by means of (53). In the case when all the l_{1s} characteristics are equal to zero the partition functions, as well as the Green functions can be given in the much simpler form [11,12,22]. The integration region in (15) is determined by the requirement of the supermodular invariance.

The investigation of the obtained multi-loop amplitudes is planned to perform in an another place. In this paper we only touch shortly the divergency problem.

Owing to the supermodular invariance one can exclude from this region of the integration in (15) those domains where some of the Schottky group multipliers k are near to unity: $k \approx 1$. Indeed, modulo of supermodular transformations, these domains are equivalent to those where some of k_j are small: $k_j \approx 0$. At $k_j \to 0$ we see from eq.(132) and (141) that $Z_L^{(n)}(\{q_{N_s}\}) \sim k_j^{-1}$ for $l_{1j} = 1/2$ and $Z_L^{(n)}(\{q_{N_s}\}) \sim k_j^{-3/2}$ for $l_{1j} = 0$. However, in the sum (15) over L the above singularity $k_j^{-3/2}$ is reduced to k_j^{-1}. Besides, we have the factor $(\ln|k|)^{-5}$ due to the non-holomorphic factor in the partition functions (90). As the result, the integral over k_j in(15) appears to be finite at $k_j \to 0$.

Nevertheless, the problem of the finiteness of the considered theory needs a further study. It follows from (132) that, beside the above singularities at $k_j \to 0$, every Z_L has also the singularities at $u_j - v_j \to 0$. One can interpret the above limit as the moving of the j-handle

away from the others. The contribution to A_n from the region where $u_j - v_j \to 0$ appears to be proportional to

$$\int \frac{d(Reu_j)d(Rev_j)d(Imu_j)d(Imv_j)d\mu_j d\nu_j d\overline{\mu}_j d\overline{\nu}_j}{|u_j - v_j - \mu_j \nu_j|^2} Z_{n-1} A_1 \tag{149}$$

where Z_{n-1} is the genus-(n-1) vacuum amplitude and $A1$ is the genus-1 amplitude. One can see that the integral (149) has uncertainty, if $Z_n \neq 0$ for all $n > 1$. The uncertainties of the same type arise also from the other regions, which correspond to the moving of the handles away from each other. The equality $Z_n = 0$ is expected [9], if the discussed theory possesses the space-time supersymmetry, as well as the world-sheet one, but till now we do not know an explicit proof of the statement that the Ramond-Neveu-Schwarz superstring really possesses the space-time supersymmetry. This problem, as well as the divergency problem in the Ramond-Neveu-Schwarz superstring theory is planned to be consider in an another paper.

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A Degree-2 polynomials in (z, θ)

The $Y_{b,k}$, $Y_{b,u}$ and $Y_{b,\mu}$ polynomials can be written down as

$$Y_{b,k_s}(t) = \frac{Y^{(0)}_{b,k_s}(t_s)}{Q^2_{\tilde{\Gamma}_s}(t_s)}, \quad Y_{b,u_s}(t) = \frac{Y^{(0)}_{b,u_s}(t_s)(u_s - v_s)}{Q^2_{\tilde{\Gamma}_s}(t_s)(u_s - v_s - \mu_s \nu_s)} + \frac{(\mu_s - \nu_s) Y^{(0)}_{b,\mu_s}(t_s)}{Q^2_{\tilde{\Gamma}_s}(t_s)(u_s - v_s)}$$
$$\text{and} \qquad Y_{b,\mu_s}(t) = \frac{(u_s - v_s - \mu_s \nu_s/2)) Y^{(0)}_{b,\mu_s}(t_s)}{Q^2_{\tilde{\Gamma}_s}(t_s)(u_s - v_s)} + \frac{\mu_s Y^{(0)}_{b,u_s}(t_s)}{Q^2_{\tilde{\Gamma}_s}(t_s)} \tag{150}$$

where both the $\tilde{\Gamma}_s$ mapping and $t_s = t_s(t)$ are defined in Sec.III by eq.(23). Furthermore,

$$Y^{(0)}_{b,k_s}(t) = \frac{(z - u_s)(z - v_s)}{k_s(u_s - v_s)}, \quad Y^{(0)}_{b,u_s}(t) = \frac{(1 - k_s)(z - v_s)^2}{k_s(u_s - v_s)^2}$$
$$\text{and} \quad Y^{(0)}_{b,\mu_s}(t) = \frac{2\theta(1 - \sqrt{k}_s)(z - v_s)}{\sqrt{k}_s(u_s - v_s)}. \tag{151}$$

One can see from (A1) that in the $\mu_s = \nu_s = 0$ case the discussed polynomials are reduced to $Y^{(0)}_{b,k_s}$, $Y^{(0)}_{b,u_s}$ and $Y^{(0)}_{b,\mu_s}$, respectively. The Y_{b,v_s} and Y_{b,ν_s} polynomials can be obtained from Y_{b,u_s} and Y_{b,μ_s}, respectively, by the replacements $k \to 1/k, u \to v, v \to u, \mu \to \nu$ and $\nu \to \mu$.

Every degree-2 polynomial $P(z, \theta)$ can be expanded over the above Y_{b,N_s} as

$$P(z, \theta) = \sum_{N_s} Y_{b,N_s} A_{N_s} \tag{152}$$

where the A_{N_s} factors are given by [12] (the index s is omitted):

$$A_k = \frac{k}{2}(u-v-\mu\nu)\partial_z^2 P(z,\theta) - \frac{k[P(v+\mu\nu,\nu)+P(u+\nu\mu,\mu)]}{(u-v-\mu\nu)},$$
$$A_u = \frac{k}{1-k}\left(1-\frac{\mu\nu}{\sqrt{k}(u-v)}\right)P(u,-\frac{1-\sqrt{k}}{2\sqrt{k}}\mu), \tag{153}$$
$$A_\mu = \frac{\sqrt{k}}{2(1-\sqrt{k})}\left(1-\frac{\mu\nu}{\sqrt{k}(u-v)}\right)\partial_\theta P(u+\theta\mu,\theta) - \frac{(\mu-\nu)A_u}{\sqrt{k}(u-v)}.$$

The A_v and A_ν values can be obtained from A_u and A_μ, respectively, by the replacements $k \to 1/k, u \to v, v \to u, \mu \to \nu$ and $\nu \to \mu$.

The $Y_{b,N_s}(t)$ and $Y_{a,N_s}(t)$ polynomials satisfy the following relation:

$$Y_{b,N_s}(t) + Q^2_{\Gamma_{b,s}}(t)Y_{a,N_s}(t^b_s) = Y_{a,N_s}(t) + Q^2_{\Gamma_{a,s}}(t)Y_{b,N_s}(t^a_s). \tag{154}$$

To prove eq.(A5) we replace t by t^a_s in the first of eqs.(39) and t by t^b_s in the second of them. For both $G_{gh}(t^a_s,t')$ and $G_{gh}(t^b_s,t')$ appearing on the right side of (39) we again use (39). The resulting left sides in both equations in (39) appear to be the same because $\Gamma_{a,s}$ commutate with $\Gamma_{b,s}$. It leads to (A5).

From (40) and using that $D(t)g^p_s = Q^{-1}_{\Gamma_{p,s}}(t)\gamma^p_s, D(t)\gamma^p_s = Q^{-1}_{\Gamma_{p,s}}(t)$ and that $\partial_z = D(t)D(t)$, one obtain both $\partial_z Y_{p,N_s}(t)$ and $D(t)Y_{p,N_s}(t)$ as

$$\partial_z Y_{p,N_s}(t) = -[D(t)\ln Q_{\Gamma_{p,s}}(t)]D(t)Y_{p,N_s} + 2[\partial_z \ln Q_{\Gamma_{p,s}}(t)]Y_{p,N_s}(t) - 2\partial_{N_s}\ln Q_{\Gamma_{p,s}}(t)$$
$$D(t)Y_{p,N_s}(t) = 2Q^{-1}_{\Gamma_{p,s}}(t)[D(t)Q_{\Gamma_{p,s}}(t)]Y_{p,N_s}(t) + 2Q_{\Gamma_{p,s}}(t)\partial_{N_s}\gamma^{(p)}_s(\theta,z). \tag{155}$$

B Green function in the boson string theory

In (44) the boson Green function $R_b(z,z')$ is given by [25]

$$R_b(z,z') = \sum_\Gamma \ln\left(\frac{[z-g_\Gamma(z')][-c_\Gamma z^{(o)}+a_\Gamma]}{[-c_\Gamma z+a_\Gamma][z^{(o)}-g_\Gamma(z^{(1)})]}\right) \tag{156}$$

$z^{(o)}$ and $z^{(1)}$ being arbitrary constants. Furthermore, [14] periods $J_{(o)s}(z)$ of the above $R_b(z,z')$ are given by

$$J_{(o)s}(z) = {\sum_\Gamma}' \ln\frac{z-g_\Gamma(u_s)}{z-g_\Gamma(v_s)} \quad , \tag{157}$$

u_s v_s being the fixed points of the Γ_s transformation. In (B8) the summation is performed over all the group products except those with the rightmost to be a power of the above Γ_s. The period matrix $\omega^{(o)}$ turns out [14,26] to be

$$\omega^{(o)}_{sp} = {\sum_\Gamma}'' \ln\frac{[u_s-g_\Gamma(u_p)][v_s-g_\Gamma(v_p)]}{[u_s-g_\Gamma(v_p)][v_s-g_\Gamma(u_p)]} + \delta_{ps}\ln k_s \quad . \tag{158}$$

In (B9) the summation is performed over all Γ except those that have the leftmost to be a power of Γ_s, the rightmost being a power Γ_r. Besides, $\Gamma \neq I$, if $s = p$.

C Integral conditions

To derive both (50) and (52) we consider the $F_{1/2,s}(t)$ function to be 1/2-supertensor under both $\Gamma_{a,s}$ and $\Gamma_{b,s}$ mappings and prove the following relation:

$$\int\limits_{C_s} F_{1/2,s}(t)dtK_s^{(1)}(t,t') = -\int\limits_{C_b^{(s)}} F_{1/2,s}(t)dt2\pi i\eta_s^{(1)}(t') + \int\limits_{C_b^{(s)}} F_{1/2,s}(t)dt\varphi_s(t)f_s(t') \tag{159}$$

where $dt = d\theta dz/2\pi i$ and the C_s contour surrounds both $C_s^{(-)}$ and $C_s^{(+)}$ circles (26) together with the $\tilde{C}_s$ cut arising for $l_{1s} \neq 0$. To prove (C10) it is convenient to use the t_s supercoordinates (23) instead of t to be variables of the integration. Moreover, we choose the intersect $z_s^{(+)} = \tilde{C}_s \bigcap \hat{C}_s^{(+)}$ as $z_s^{(+)} = g_s^{(b)}(z_s^{(-)})$ where $z_s^{(-)} = \tilde{C}_s \bigcap \hat{C}_s^{(-)}$ and $z_s \to g_s^{(b)}$ is the Schottky transformation corresponding to 2π-twist about B_s-cycle. The integral along the $\tilde{C}_s$ cut disappears owing to (29). The integral along $\hat{C}_s^{(+)}$ is reduced to the integral along the $\hat{C}_s^{(-)}$ contour by the Schottky transformation (17). After the above integral to be summed with the integral along $\hat{C}_s^{(-)}$, the right side of (C10) arises.

Taking $F_{1/2,s} = f_s$, one obtains that the left side of (C10) is equal to f_s owing to the first of eqs.(51). So two equations in (50) appear. To prove the third relation in (50) we choose $F_{1/2,s}(t')$ to equal to $K_s^{(1)}(t,t')$. In this case the integral along C_s-contour also can be reduced to the integral along $C_b^{(s)}$, but the integrand appears to be different from that given in (C10) because $K_s^{(1)}(t,t')$ has periods about B_s-cycle. Owing to the second equation of eqs.(51), one obtains a number of relations, both the third equation of eqs.(50) and (52) being among of them.

As far as the $\Gamma_{a,s}$ and $\Gamma_{b,s}$ mappings are the same for both $K_s^{(1)}$ and K, the above consideration can be applied also to the integration over t_1 along the C_s contour of $f_s(t)K(t_1,t')$. In this case the first term on the right side of (C10) contains η_s instead of $\eta_s^{(1)}$. The last term disappears because the fermion zero mode is absent in the case discussed. So the last of eqs.(50) remains to be true, if one takes K instead of $K_s^{(1)}$. In addition, one concludes that

$$\int\limits_{C_s} f_s d\theta dz K_(t,t') = 0 \quad \text{and} \quad \int\limits_{C_s} K(t',t)d\theta dz\varphi_s(t) = 0. \tag{160}$$

The second eq.(C11) follows from the first one because $f_s(t) = D(t)\varphi_s(t)$ and $D(t)R_s^{(1)}(t,t') = K_s^{(1)}(t',t)$.

To prove eq.(53) we write down the sum over r in (53) as

$$-\sum_{r=1}^{n}\int\limits_{C_r} K_{(o)}(t,t_1)dt_1K_r^{(1)}(t_1,t_2)dt_2K(t_2,t') + \int\limits_{C_r} K_{(o)}(t,t_1)dt_1K_{(o)r}^{(1)}(t_1,t_2)dt_2K(t_2,t') \tag{161}$$

where one can think that in the integral over z_1 the pole at $z_1 = z_2$ in both $K_{(o)r}^{(1)}(t_1,t_2)$ and $K_r^{(1)}(t_1,t_2)$ is surrounded by C_r-contour. When the integral over z_1 in the second term is reduced to $C_b^{(r)}$, the contribution of the above pole appears to be

$$\sum_{r=1}^{n}\int\limits_{C_r} K_{(o)}(t,t_2)dt_2K(t_2,t') = K(t,t') - K_{(o)}(t,t'). \tag{162}$$

The rest of (C12) together with to two other terms on the right side of (53) turns out to be equal to zero owing to (C11). It proves eq.(53).

To prove (54) and (55) we do as in proving (C10). Only the integral along the cut contributes into the right side of (54). The eq.(55) can be obtained also by the calculation of the period about B_s-cycle of J_s given by (54).

The kindred statements given in Sec.V can be derived in the quite similar manner. For this aim we consider the $F_{3/2,s}(t)$ function to be 3/2-supertensor under both $\Gamma_{a,s}$ and $\Gamma_{b,s}$ mappings and prove the following relation:

$$\int_{C_s} F_{3/2,s}(t)dt\tilde{G}(t,t') = -\sum_{\{N_s\}}\left[\int_{C_b^{(s)}} F_{3/2,s}(t)dtF_{b,N_s}(t) + \int_{C_a^{(s)}} F_{3/2,s}(t)dtF_{a,N_s}(t)\right]\Delta_{N_s}(t') \tag{163}$$

where the $C_b^{(s)}$ contour and the $C_a^{(s)}$ path are the same as in (66). mappings. Moreover, $\tilde{G}_s(t,t')$ is the Green function that transforms under the $(\Gamma_{a,s},\Gamma_{b,s})$ mappings by (63) with $F_{b,N_s}(t)$ superfunctions instead of the polynomials and $\Delta_{N_s}(t')$ to be 3/2-supertensors instead of $\chi_{N_s}(t')$ in (63).

Being applied for $F_{3/2,s}(t) = G(t_1,t)$ together with $\tilde{G}(t,t')$ to be $G(t,t')$, one obtains that the left side of (C14) is equal to zero that leads to eq.(66). To prove (73) one uses (C14) for $F_{3/2,s}(t) = \theta G_f(z_1,z)$ and $\tilde{G}(t,t') = \theta G_{(\sigma)}(z,z')$. To prove (76) one uses (C14) for $F_{3/2,s}(t) = S_\sigma(t_1,t)$ and $\tilde{G}(t,t') = G(t,t')$. To prove (79) one uses (C14) for $F_{3/2,s}(t) = \theta G_{(\sigma)}(z_1,z)$ and $\tilde{G}(t,t') = \theta S_{(f)\sigma}(z,z')$. Moreover, employing $F_{3/2,s}(t) = S^{(1)}_{\sigma,s}(t_1,t)$ and $\tilde{G}(t,t') = S^{(1)}_{\sigma,s}(t,t')$, one obtains that

$$\int_{C_b^{(s)}} S_\sigma(t,t')\frac{d\theta' dz'}{2\pi i}\hat{Y}^{(1)}_{\sigma,N_s}(t') = 0. \tag{164}$$

In Sec.VI the above method is used to derive the integral relations (87) for $\tilde{\chi}_{N_s}(t)$ and eqs.(88) for $\chi_{N_s}(t)$ and $\Psi_{\sigma,N_s}(t)$.

D Integral representations

To obtain eq.(91) we represent $\partial_{N_r}J_m(t)$ as it follows

$$\partial_{N_r}J_m(t) = -\int_{C(z)} K(t,t')dt'\partial_{N_r}J_m(t') = \sum_r\int_{C_r} K(t,t')dt'\partial_{N_r}J_m(t') \tag{165}$$

where $K(t,t')$ is defined by eq.(36) and $dt = d\theta dz/2\pi i$. The integrals along C_r are reduced to the integrals along both the $C_a^{(r)}$ and $C_b^{(r)}$ paths defined by (66). For the calculation of the periods of $\partial_{N_r}J_m(t)$ we use (92). We use also the second of eqs.(50) for $K(t,t')$. The resulting relations are found to be

$$\partial_{N_r}J_m(t) = -\pi i\sum_{p=a,b}\int_{C_p^{(r)}} K(t,t')dt\{[D(t')\eta_m(t')]Y_{N_r}(t') - D(t')[\eta_m(t')Y_{N_r}(t')]\} \tag{166}$$

where $\eta_m(t')$ are the half-forms (37). Calculating the periods about B_s-cycle of the left and right sides of (D17), we obtain the integral representation for $\partial_{N_r}\omega_{mn}$ in the following form:

$$\partial_{N_r}\omega_{mn} = -\pi i \sum_{p=a,b} \int\limits_{C_p^{(r)}} \eta_n(t)dt\{[D(t')\eta_m(t')]Y_{N_r}(t') - D(t')[\eta_m(t')Y_{N_r}(t')]\}. \tag{167}$$

The last term in (D18) is integrated by parts. Out integral terms disappear owing to (A5). And we obtain eq.(91).

To obtain another useful relations, one can integrate along the C_r contour the quantities $f_r(t')\partial_{N_r}K_r^{(1)}(t',t)$ and $f_r(t)\partial_{N_r}\varphi_r(t)$. The above integrals are equal to zero because both the integrands have no singularities outside of the C_r contour. In this case, using (92), we obtain that

$$\sum_{p=a,b}\int\limits_{C_p^r} f_r(t')dt'E_{N_r}^{(0)}\left(K_r^{(1)}(t',t)\right) - \int\limits_{C_b^r} f_r(t')dt'\left[E_{N_r}^{(0)}\left(\varphi_r(t')f_r(t)\right) - \partial_{N_r}[\varphi_r(t')f_r(t)]\right] = 0$$

$$\text{and} \quad \sum_{p=a,b}\int\limits_{C_p^r} f_r(t')dt'E_{N_r}^{(0)}\left(\varphi_r(t')\right) = 0 \tag{168}$$

where $E_{N_r}^{(0)}\left(F(t,t')\right)$ is defined by (93) at $q=0$.

The value of $\partial_{N_r}K_r^{(1)}(t,t')$ for the even genus-1 spin structures is given in Sec.VI. For the odd genus-1 spin structure the same method gives that

$$\partial_{N_r}K_r^{(1)}(t,t') = \int\limits_{C_a^r} K_r^{(1)}(t,t_1)dt_1E_{N_r}^{(0)}\left(K_r^{(1)}(t_1,t')\right) - \varphi_r(t)\int\limits_{C_b^r} f_r(t_1)dt_1\partial_{N_r}K_r^{(1)}(t_1,t') +$$
$$\int\limits_{C_b^r}\left[K_r^{(1)}(t,t_1) + \varphi_r(t)f_r(t_1)\right]dt_1\left[\partial_{N_r}\left(\varphi_r(t_1)f_r(t')\right) + E_{N_r}^{(0)}\left(K_r^{(1)}(t_1,t') - \varphi_r(t_1)f_r(t')\right)\right] \tag{169}$$

The same method allows also to obtain the following representation for $\partial_{N_r}\varphi_r(t)$:

$$\partial_{N_r}\varphi_r(t) = \sum_{p=a,b}\int\limits_{C_p^r} K_r^{(1)}(t,t_1)dt_1E_{N_r}^{(0)}\left(\varphi_r^{(1)}(t_1)\right) + \varphi_r(t)\int\limits_{C_b^r} f_r(t_1)E_{N_r}^{(0)}\left(\varphi_r^{(1)}(t_1)\right) -$$
$$\varphi_r(t)\int\limits_{C_b^r} f_r(t_1)\partial_{N_r}\varphi_r(t_1). \tag{170}$$

To obtain $\partial_{M_r}S_{\sigma,r}^{(1)}(t,t')$ we integrate along the infinitesimal $C(z_1)$ contour the quantities $S_{\sigma,r}^{(1)}(t,t_1)\partial_{M_r}S_{\sigma,r}^{(1)}(t_1,t')$ and we deform the contour as it was explained above. The result is

$$\partial_{M_r}S_{\sigma,r}^{(1)}(t,t') = \sum_{p=a,b}\int\limits_{C_p^r} S_{\sigma,r}^{(1)}(t,t_1)dt_1E_{M_r}^{(-2)}\left(S_{\sigma,r}^{(1)}(t_1,t')\right) + \sum_{N_r}\int\limits_{C_b^r} S_{\sigma,r}^{(1)}(t,t_1)dt_1 \times$$
$$E_{M_r}^{(-2)}\left(S_{\sigma,r}^{(1)}(t_1,t')\right) - \sum_{N_r}\int\limits_{C_b^r} S_{\sigma,r}^{(1)}(t,t_1)dt_1\partial_{M_r}[\hat{Y}_{\sigma,N_r}^{(1)}(t_1)\Psi_{\sigma,N_r}^{(1)}(t')] \tag{171}$$

where $E^{(-2)}_{M_r}(F(t,t'))$ is defined by (93) at $q = -2$. Furthermore, one could integrate over t along the above $C(z_1)$ contour the quantities $\chi_{N_s}(t)\partial_{M_r}S_\sigma(t,t')$. In this case one obtains that

$$\sum_{p=a,b}\int_{C_p^r}\chi_{N_s}(t)dtE^{(-2)}_{M_r}(S_\sigma(t,t')) + \sum_{N_r}\int_{C_b^r}\chi_{N_s}(t)dt\,\times$$

$$E^{(-2)}_{M_r}(S_\sigma(t,t')) - \sum_{N_r}\int_{C_b^r}\chi_{N_s}(t)dt\partial_{M_r}[\hat{Y}^{(1)}_{\sigma,N_r}(t_1)\Psi_{\sigma,N_r}(t')] = 0. \tag{172}$$

E Derivatives of the Schottky group multipliers.

To obtain the desired values of $\partial_{u_s}k, \partial_{v_s}k$ and of $\partial_{k_s}k$, we write down the Schottky group product g having the multiplier to be k as follows

$$g = \prod_{j=1}^{q} g_{(j)}g_s^{n_j} \tag{173}$$

where $g_s^{n_j}$ are powers of the given g_s basic Schottky group element and the $g_{(j)}$ group products do not contain g_s. In this case the $\delta g(z)$ alteration of $g(z)$ with respect to variations δu_s, δv_s and δk_s of k_s, u_s and v_s is given by

$$Q_g(z)^2\delta g(z) = \sum_{j=1}^{q} Q^2_{g(>j)}(z)[Y_{k_s}(z_j)\delta k_s + \frac{(1-k_s^{n_j})k_s}{k_s^{n_j}((1-k_s)}Y_{u_s}(z_j)\delta u_s + \frac{(1-k_s^{n_j})}{((1-k_s)}Y_{v_s}(z_j)\delta v_s] \tag{174}$$

where $Q_g^{-2}(z) = \partial_z g(z)$, $z_j = g(>j;z)$ and the $g(>j)$ mapping is defined as

$$g(>j) = \prod_{p=j+1}^{q} g_{(p)}g_s^{n_p} \tag{175}$$

with $g(>q) = I$. Furthermore, Y_{u_s}, Y_{v_s} and Y_{k_s} in (E25) are none other than the Y_{b,N_s} polynomials defined by (40) at zero odd parameters. On the other hand, from (40) it is follows also that

$$Q_g(z)^2\delta g(z) = Y_k(z)\delta k + Y_u(z)\delta u + Y_v(z)\delta v \tag{176}$$

where $Y_u(z), Y_v(z)$ and $Y_k(z)$ are the degree-2 polynomials corresponding to the $z \to g(z)$ transformation. Moreover, the right side of (E25) can be expanded over the above $Y_u(z), Y_v(z)$ and $Y_k(z)$ polynomials by means of eqs.(A3). In this case one obtains both $\delta u, \delta v$ and δk in the terms of $\delta u_s, \delta v_s$ and δk_s. So, one can obtain the derivatives with respect to u_s, v_s and k_s of every u, v and of k.

References

[1] P. Ramond, Phys.Rev. D3 (1971) 2415.
A. Neveu and J.H. Schwarz, Nucl.Phys. B31 (1971) 86.

[2] F. Gliozzi, D. Olive and J. Scherk, Phys.Lett. 65B (1976) 282.

[3] M.B. Green and J.S. Schwarz, Nucl.Phys. B 181 (1981) 502;
Phys. Lett. B 109 (1982) 444; Phys. Lett. B 136 (1984) 367.

[4] M.B. Green, J.H. Schwarz and E. Witten, Superstring Theory, vols.I and II (Cambridge Univesity Press, England, 1987).

[5] J. Scherk and J.H. Schwarz, Nucl. Phys. B 81 (1974) 118; Phys. Lett. B 52 (1974) 347.

[6] M.B.Green and J.H.Schwarz, Phys.Lett. B 149(1984) 117.
M.B.Green and J.H.Schwarz, Phys.Lett. B 151(1985) 21.
D.J. Gross, J.A. Harvey, E. Martinec and R. Rohm, Nucl. Phys. B 156 (1985) 253; Nucl. Phys. B 267 (1986) 75.

[7] A.A. Belavin and V.G. Knizhnik, Phys.Lett. B 168 (1986) 201; ZhETF 91 (1986) 364.

[8] N. Berkovits, Nucl. Phys. B408 (1993) 43.

[9] E. Martinec, Phys. Lett. B 171 (1986) 189.

[10] G.S. Danilov, Sov. J. Nucl. Phys. 49 (1989) 1106 [Jadernaja Fizika 49 (1989) 1787].

[11] G.S. Danilov, Phys. Lett. B 257 (1991) 285;

[12] G.S. Danilov, Sov. J. Nucl. Phys. 52 (1990) 727 [Jadernaja Fizika 52 (1990) 1143].

[13] E. Verlinde and H. Verlinde, Phys. Lett. B 192 (1987) 95.

[14] E. Martinec, Nucl. Phys. B 281 (1986) 157.

[15] J. Atick and A. Sen, Nucl.Phys. B. 296 (1988) 157;
J. Atick, J. Rabin and A. Sen, Nucl. Phys. B 299 (1988) 279.

[16] G. Moore and A. Morozov, Nucl. Phys. B 306 (1988) 387; A. Yu. Morozov, Teor. Mat. Fiz. 81 (1989) 24.

[17] N. Seiberg and E. Witten, Nucl.Phys. B276 (1986) 272.

[18] E. Verlinde and H. Verlinde, Nucl. Phys. B 288 (1987) 357.

[19] O. Lechtenfeld and A. Parkes, Nucl.Phys. B 332 (1990) 39.

[20] S. Mandelstam, Phys.Lett. B 277 (1992) 82.

[21] M.A. Baranov and A.S. Schwarz, Pis'ma ZhETF 42 (1985) 340 [JETP Lett. 49 (1986) 419]; D. Friedan, Proc. Santa Barbara Workshop on Unified String theories, eds. D. Gross and M. Green (World Scientific, Singapure, 1986).

[22] P. Di Vecchia, K. Hornfeck, M. Frau, A. Ledra and S. Sciuto, Phys. Lett. B 211 (1988) 301.

[23] J.L. Petersen, J.R. Sidenius and A.K. Tollsten, Phys Lett. B 213 (1988) 30; Nucl. Phys. B 317 (1989) 109.

[24] C. Lovelace, Phys. Lett. B 32 (1970) 703; V. Alessandrini, Nuovo Cim. 2A (1971) 321; V. Alessandrini and D. Amati, Nuovo Cim. 2A (1971) 793.

[25] D. Friedan, E. Martinec and S. Schenker, Nucl. Phys. B 271 (1986) 93.

[26] P. Di Vecchia, M. Frau, A. Ledra and S. Sciuto, Phys. Lett. B 199 (1987) 49.

[27] L. Hodkin, JPG, 6 (1989) 333.

[28] G.S. Danilov, PNPI-1872 (May 1993), hep/th 9305029.

[29] G.S. Danilov, JETP Lett. 58 (1993) 796 [Pis'ma JhETF 58 (1993) 790.]

[30] G.S. Danilov, Jadernaja Fizika 57 (1994) 159.

[31] A.M. Polyakov, Phys. Lett. B 103 (1981) 210; Phys. Lett. B 103 (1981) 207.

[32] A.A. Rosly, A.S. Schwarz and A.A. Voronov, Commun. Math. Phys. 119 (1986) 129.

[33] D.A. Leites, Usp. Mat. Nauk, 35 (1980) 1.